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# Mathematics for Data Science

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Chapter 3

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# Linear Independence and Dependence of Vectors

## Linear Independence. Rank of a Matrix. Vector Space

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## Inverse of a Matrix

Given any set of  $m$  vectors  $\mathbf{a}_{(1)}, \dots, \mathbf{a}_{(m)}$  (with the same number of components), a linear combination of these vectors is an expression of the form :

$$c_1 \mathbf{a}_{(1)} + c_2 \mathbf{a}_{(2)} + \dots + c_m \mathbf{a}_{(m)}$$

where  $c_1, c_2, \dots, c_m$  are any scalars. Now consider the equation

$$c_1 \mathbf{a}_{(1)} + c_2 \mathbf{a}_{(2)} + \dots + c_m \mathbf{a}_{(m)} = \mathbf{0}. \quad (1.1)$$





# Rank of a Matrix

## Definition

The **rank** of a matrix **A** is the maximum number of linearly independent row vectors of **A**. Its is denoted by rank **A**

## Rank

The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad (1.2)$$

has rank 2, cause Example 1 shows that the first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.



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A **determinant of second order** is denoted and defined by

$$D = \det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (2.1)$$

So here we have **bars** (whereas a matrix has **brackets**).





# Second- and Third-Order Determinants

**Cramer's rule** for solving linear systems of two equations in two unknowns

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2\end{aligned}\tag{2.2}$$

is

$$\begin{aligned}x_1 &= \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_{11}a_{22} - a_{12}b_2}{D} \\ x_2 &= \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11}b_2 - a_{21}b_1}{D}\end{aligned}\tag{2.3}$$

with  $D$  as in 2.1, provided  $D \neq 0$ . The value  $D = 0$  appears for homogeneous systems with nontrivial solutions.



# Cramer's Rule for Two Equations

If

$$4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

then

$$x_1 = \frac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{84}{14} = 6$$

$$x_2 = \frac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{-56}{14} = -4$$



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A determinant of third order can be defined by

$$D = \begin{vmatrix} a_{11} & b_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad (2.4)$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \quad (2.5)$$
$$+ a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$





# Third-Order Determinants

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Note the following. The signs on the right are  $+ - +$ . Each of the three terms on the right is an entry in the first column of  $D$  times its minor, that is, the second-order determinant obtained from  $D$  by deleting the row and column of that entry; thus, for  $a_{11}$  delete the first row and first column, and so on.

If we write out the minors in 2.4, we obtain

$$\begin{aligned} D = & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} & (2.6) \\ & - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22} \end{aligned}$$



# Cramer's Rule for Linear Systems of Three Equations

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$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3\end{aligned}\tag{2.7}$$

is

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D} \quad (D \neq 0) \tag{2.8}$$



# Cramer's Rule for Linear Systems of Three Equations

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with the determinant  $D$  of the system given by 2.1 and

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Note that  $D_1, D_2, D_3$  are obtained by replacing Columns 1, 2, 3, respectively, by the column of the right sides of 2.7.





# Example

If we have :

$$2x_1 - 3x_2 + 2x_3 = -3$$

$$x_1 + 2x_2 + x_3 = 2$$

$$2x_1 - x_2 + 3x_3 = 1$$

then

$$D = 7, \quad D_1 = -14, \quad D_2 = 7, \quad D_3 = 14$$

So we have

$$x_1 = -2, \quad x_2 = 1, \quad x_3 = 2.$$





# Determinants. Cramer's Rule

Determinants were originally introduced for solving linear systems. Although impractical in computations, they have important engineering applications in eigenvalue problems, differential equations, vector algebra, and in other areas. They can be introduced in several equivalent ways. Our definition is particularly for dealing with linear systems.

A determinant of order  $n$  is a scalar associated with an  $n \times n$  (hence **square!**) matrix  $\mathbf{A} = [a_{jk}]$ , and is denoted by

$$D = \det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (2.9)$$





# Determinants: Cramer's Rule

For  $n = 1$ , this determinant is defined as

$$D = a_{11} \quad (2.10)$$

For  $n \geq 2$ , it is defined as

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \quad (2.11)$$

$$(j = 1, 2, \dots, n)$$

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \quad (2.12)$$

$$(k = 1, 2, \dots, n)$$

Here,

$$C_{jk} = (-1)^{j+k} M_{jk}$$

and  $M_{jk}$  is a determinant of order  $n - 1$ .



# Minors and Cofactors of a Third-Order Determinant

In 2.9 of the previous section the minors and cofactors of the entries in the first column can be seen directly. For the entries in the second row the minors are

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and the cofactors are  $C_{21} = -M_{21}$ ,  $C_{22} = M_{22}$  and  $C_{23} = -M_{23}$  Similarly for the third row— write these down yourself. And verify that the signs in  $C_{jk}$  form a **checkerboard pattern**

$$\begin{array}{ccc} + & - & + \\ - & + & - \end{array}$$





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This is the expansion by the first row.

$$\begin{aligned} D &= \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} \\ &= 1 \begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} \\ &= 1(12 - 0) - 3(4 + 8) + 0(0 - 6) \\ &= -12. \end{aligned}$$





# Formulas for Inverses

## Definition (Inverse of a Matrix by Determinants)

The inverse of a nonsingular  $n \times n$  matrix  $A = [a_{jk}]$  is given by

$$A^{-1} = \frac{1}{\det \mathbf{A}} [C_{jk}]^T = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

where  $C_{jk}$  is the cofactor of  $a_{jk}$  in  $\det(\mathbf{A})$

## Caution!

Note well that in  $A^{-1}$ , the cofactor  $C_{jk}$  occupies the same place as  $a_{kj}$  (not  $a_{jk}$ ) does in  $\mathbf{A}$ .





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## Definition (Inverse of a Matrix by Determinants)

in particular, the inverse of

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{is} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$





# Inverse of a $2 \times 2$ Matrix by Determinants

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Matrix**

Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix},$$

then

$$\mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$





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Let

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

, We obtain  $\det(\mathbf{A}) = 10$  and

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \quad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \quad C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \quad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \quad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2, \quad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$





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and we have solution

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$



Thank You.

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