Mathematics for Data Science

SSD23402

Chapter 9

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Modulus of Complex Numbers

Complex Numbers

A complex number is a number that has two components: a real part and an imaginary part, written as $a + bi$, where a is the real part, b is the imaginary part, and i is the imaginary unit ($i^2 = -1$).

Modulus of a Complex Number

The modulus of $z = a + bi$ is denoted by $|z|$ and is calculated as:

$$|z| = \sqrt{a^2 + b^2}$$

The modulus is always a non-negative number.

Argument of Complex Numbers

Argument of a Complex Number

The argument of $z = a + bi$ is denoted by θ and can be calculated as:

$$\theta = \arctan\left(\frac{b}{a}\right)$$

For the argument in degrees: Argument in degrees $= \theta \times \left(\frac{180}{\pi}\right)$

Relationship between Modulus and Argument

In complex geometry, a complex number z can be represented in modulus-argument form ($z = |z| \cdot \text{cis}(\theta)$), where $\text{cis}(\theta)$ is the exponential notation of the complex number given by:

$$\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$$

Example

Example: $z = 3 + 4i$

Suppose we have a complex number $z = 3 + 4i$. To calculate its modulus and argument, we proceed as follows:

1. Modulus:

$$|z| = \sqrt{3^2 + 4^2} = 5$$

2. Argument (in radians):

$$\theta = \arctan\left(\frac{4}{3}\right) \approx 0.93 \text{ radians}$$

3. Argument (in degrees):

$$\text{Argument in degrees} \approx 53.13^\circ$$

Complex Numbers Raised to Natural Powers

Natural Powers of Complex Numbers

Natural powers of complex numbers refer to raising complex numbers to positive integer exponents $(n = 1, 2, 3, \ldots)$. Complex numbers are usually in the form $z = a + bi$, where a is the real part and b is the imaginary part.

Exponentiation Operation

The exponentiation operation is performed using the following rule:

$$z^n = (a + bi)^n = a^n + \binom{n}{1} a^{n-1}(bi) - \binom{n}{2} a^{n-2}(bi)^2 + \ldots$$

In the above formula, $\binom{n}{k}$ represents the binomial coefficient, and $i^2 = -1$.

Example Questions

Question 1

Calculate z^4 if $z = 3 + 2i$.

Question 2

Calculate z^5 if $z = -1 + i$.

Solution to Question 1

Solution to Question 1

Using the complex exponentiation formula:

$$z^4 = (3+2i)^4$$

$$z^4 = 3^4 + \binom{4}{1}3^3(2i) - \binom{4}{2}3^2(2i)^2 + \binom{4}{3}3(2i)^3 - 2^4$$

$$z^4 = 81 + 216i - 216 - 32i - 16$$

$$z^4 = 65 + 184i$$

$$z^4 = 65 + 184i$$

Solution to Question 2

Solution to Question 2

Using the complex exponentiation formula:

$$z^5 = (-1+i)^5$$

$$z^5 = (-1)^5 + \binom{5}{1}(-1)^4(i) - \binom{5}{2}(-1)^3(i)^2 + \binom{5}{3}(-1)^2(i)^3 - i^5$$

$$z^5 = -1 - 5i + 10 - 10i - i$$

$$z^5 = 9 - 16i$$

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De Moivre's Theorem

Definition

De Moivre's Theorem is a formula used for raising complex numbers to powers in polar form. The formula states that for raising a complex number z to an exponent n, the result is a complex number in polar form with radius (r) raised to the nth power and the angle (θ) multiplied by n.

De Moivre's Theorem Formula

$$z^n = r^n \text{cis}(n\theta)$$

Explanation

Explanation

- z is a complex number in polar form $z = r\text{cis}(\theta)$.
- n is the exponent (an integer).
- r is the radius (magnitude) of z, i.e., $r = |z|$.
- θ is the angle (argument) of z, which can be expressed as $\theta = \text{arg}(z)$.
- $\text{cis}(\alpha)$ is the polar notation for complex numbers meaning $\text{cis}(\alpha) = \cos(\alpha) + i\sin(\alpha)$.

Benefits of De Moivre's Theorem

The formula of De Moivre's Theorem is useful for exponentiating complex numbers in polar form without converting them to Cartesian form. This simplifies calculations in many cases, especially when dealing with repeated exponentiation of complex numbers.

Problem: $(1-\sqrt{3}i)^3$

Step 1: Expressing in Complex Form

We start by expressing this complex number in the form $r\text{cis}(\theta)$.

$$z = 1 - \sqrt{3}i$$

$$r = |z| = \sqrt{1^2 + (-\sqrt{3})^2} = 2$$

$$\theta = \arctan\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

Thus, $z = 2\text{cis}\left(-\frac{\pi}{3}\right)$.

Problem: $(1-\sqrt{3}i)^3$ (continued)

Step 2: Using De Moivre's Theorem

We will use De Moivre's Theorem to calculate $(1-\sqrt{3}i)^3$.

$$(1-\sqrt{3}i)^3 = 2^3\text{cis}\left(3\left(-\frac{\pi}{3}\right)\right)$$

$$= 8\text{cis}(-\pi)$$

Step 3: Final Result

Next, we evaluate $\text{cis}(-\pi)$.

$$\text{cis}(-\pi) = \cos(-\pi) + i\sin(-\pi) = -1$$

Therefore, $(1-\sqrt{3}i)^3 = 8\cdot(-1) = -8$.

Problem: $(\sqrt{3} - i)^{-6}$

Step 1: Expressing in Complex Form

We start by expressing this complex number in the form $r\text{cis}(\theta)$.

$$z = \sqrt{3} - i$$
$$r = |z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$
$$\theta = \arctan\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

Thus, $z = 2\text{cis}\left(-\frac{\pi}{6}\right)$.

Problem: $(\sqrt{3} - i)^{-6}$ (continued)

Step 2: Using De Moivre's Theorem

We will use De Moivre's Theorem to calculate $(\sqrt{3} - i)^{-6}$.

$$(\sqrt{3} - i)^{-6} = 2^{-6}\text{cis}\left(-6\left(-\frac{\pi}{6}\right)\right)$$

$$= 2^{-6}\text{cis}(\pi)$$

Step 3: Final Result

Next, we evaluate cis(π).

$$\text{cis}(\pi) = \cos(\pi) + i\sin(\pi) = -1$$

Therefore, $(\sqrt{3} - i)^{-6} = 2^{-6} \cdot (-1) = -\frac{1}{64}$.

Conclusion

Conclusion

De Moivre's Theorem is a formula that simplifies the exponentiation of complex numbers in trigonometric form. It allows us to calculate the powers of complex numbers more easily.

Roots of Complex Numbers

Definition of Roots

The root of z is to find a complex number w such that $w^n = z$. There are n roots w_k, with $k = 0, 1, 2, \ldots, n-1$.

Calculating Roots

To find the root $z^{1/n}$, we use De Moivre's Theorem:

$$z^{1/n} = \sqrt[n]{|z|}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$$

Where $k = 0, 1, 2, \ldots, n-1$.

Example Problems

Example Problem 1

Calculate the fourth root of $z = 16\text{cis}(60°)$.

Example Problem 2

Calculate the third root of $z = 8\text{cis}(-45°)$.

Solutions

Example Problem 1

Using the root formula:

$$w_k = \sqrt[4]{16}\left(\cos\left(\frac{60^\circ + 2\pi k}{4}\right) + i\sin\left(\frac{60^\circ + 2\pi k}{4}\right)\right)$$

For $k = 0, 1, 2, 3$, we have four roots:

$$k = 0 : \sqrt[4]{16}\left(\cos(15^\circ) + i\sin(15^\circ)\right)$$

$$k = 1 : \sqrt[4]{16}\left(\cos(105^\circ) + i\sin(105^\circ)\right)$$

$$k = 2 : \sqrt[4]{16}\left(\cos(195^\circ) + i\sin(195^\circ)\right)$$

$$k = 3 : \sqrt[4]{16}\left(\cos(285^\circ) + i\sin(285^\circ)\right)$$

Solution (continued)

Example Problem 2

Using the root formula:

$$w_k = \sqrt[3]{8}\left(\cos\left(\frac{-45^\circ + 2\pi k}{3}\right) + i\sin\left(\frac{-45^\circ + 2\pi k}{3}\right)\right)$$

For $k = 0, 1, 2$, we have three roots:

$$k = 0 : \sqrt[3]{8}\left(\cos(-15^\circ) + i\sin(-15^\circ)\right)$$
$$k = 1 : \sqrt[3]{8}\left(\cos(105^\circ) + i\sin(105^\circ)\right)$$
$$k = 2 : \sqrt[3]{8}\left(\cos(225^\circ) + i\sin(225^\circ)\right)$$

Problem: $z = (-16)^{1/4}$

Step 1

To solve $z = (-16)^{1/4}$ using the roots of complex numbers, we first express z in polar form and then calculate the roots.

$$z = (-16)^{1/4} = \sqrt[4]{16} \cdot \sqrt[4]{-1}$$

We know that $\sqrt[4]{16} = 2$ because the fourth root of 16 is 2. Next, $\sqrt[4]{-1}$ represents the fourth roots of -1.

Step 2: Complex Roots

The formula for the fourth roots of -1 in polar form is as follows:

$$\sqrt[4]{-1} = \sqrt[4]{1}\text{cis}\left(\frac{\pi + 2k\pi}{4}\right)$$

where $k = 0, 1, 2, 3$ represents the root index.
Now we can calculate the values of z:

$$z_0 = 2 \cdot \text{cis}\left(\frac{\pi + 2 \cdot 0 \cdot \pi}{4}\right) = 2 \cdot \text{cis}\left(\frac{\pi}{4}\right)$$

$$z_1 = 2 \cdot \text{cis}\left(\frac{\pi + 2 \cdot 1 \cdot \pi}{4}\right) = 2 \cdot \text{cis}\left(\frac{3\pi}{4}\right)$$

$$z_2 = 2 \cdot \text{cis}\left(\frac{\pi + 2 \cdot 2 \cdot \pi}{4}\right) = 2 \cdot \text{cis}\left(\frac{5\pi}{4}\right)$$

$$z_3 = 2 \cdot \text{cis}\left(\frac{\pi + 2 \cdot 3 \cdot \pi}{4}\right) = 2 \cdot \text{cis}\left(\frac{7\pi}{4}\right)$$

Continuation

Given:

$$2 \cdot \text{cis}\left(\frac{\pi}{4}\right)$$

The first step is to find the exact value of cis $\left(\frac{\pi}{4}\right)$.

$$\text{cis}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

To find the exact values of $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$, we use common trigonometric values.

$$\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Therefore,

$$\text{cis}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

Continuation

Next, we can multiply the result by 2:

$$2 \cdot \text{cis}\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

Now, we can simplify it:

$$2 \cdot \text{cis}\left(\frac{\pi}{4}\right) = \sqrt{2} + \sqrt{2}i$$

So, the result of $2 \cdot \text{cis}\left(\frac{\pi}{4}\right)$ is $\sqrt{2} + \sqrt{2}i$.

Conclusion

Conclusion

The roots of complex numbers can be calculated using De Moivre's Theorem. There are n different roots for a complex number, and we can calculate them by varying the value of k from 0 to $n - 1$.

Thank You.

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