# CHAPTER 1 SPEAKING MATHEMATICALLY

*Therefore O students study mathematics and do not build without foundations.* —Leonardo da Vinci (1452–1519)

The aim of this book is to introduce you to a mathematical way of thinking that can serve you in a wide variety of situations. Often when you start work on a mathematical problem, you may have only a vague sense of how to proceed. You may begin by looking at examples, drawing pictures, playing around with notation, rereading the problem to focus on more of its details, and so forth. The closer you get to a solution, however, the more your thinking has to crystallize. And the more you need to understand, the more you need language that expresses mathematical ideas clearly, precisely, and unambiguously.

This chapter will introduce you to some of the special language that is a foundation for much mathematical thought, the language of variables, sets, relations, and functions. Think of the chapter like the exercises you would do before an important sporting event. Its goal is to warm up your mental muscles so that you can do your best.

## 1.1 Variables

A variable is sometimes thought of as a mathematical "John Doe" because you can use it as a placeholder when you want to talk about something but either (1) you imagine that it has one or more values but you don't know what they are, or (2) you want whatever you say about it to be equally true for all elements in a given set, and so you don't want to be restricted to considering only a particular, concrete value for it. To illustrate the first use, consider asking

> Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

In this sentence you can introduce a variable to replace the potentially ambiguous word "it":

> Is there a number $x$ with the property that $2x + 3 = x^2$?

The advantage of using a variable is that it allows you to give a temporary name to what you are seeking so that you can perform concrete computations with it to help discover its possible values. To emphasize the role of the variable as a placeholder, you might write the following:

> Is there a number $\square$ with the property that $2 \cdot \square + 3 = \square^2$?

The emptiness of the box can help you imagine filling it in with a variety of different values, some of which might make the two sides equal and others of which might not.

In this sense, a variable in a computer program is similar to a mathematical variable because it creates a location in computer memory (either actual or virtual) into which values can be placed.

To illustrate the second use of variables, consider the statement

> No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

In this case introducing a variable to give a temporary name to the (arbitrary) number you might choose enables you to maintain the generality of the statement, and replacing all instances of the word "it" by the name of the variable ensures that possible ambiguity is avoided:

> No matter what number $n$ might be chosen, if $n$ is greater than 2, then $n^2$ is greater than 4.

### **Example 1.1.1** Writing Sentences Using Variables

Use variables to rewrite the following sentences more formally.

a. Are there numbers with the property that the sum of their squares equals the square of their sum?

b. Given any real number, its square is nonnegative.

**Solution**

**Note** In part (a) the answer is yes. For instance, $a = 1$ and $b = 0$ would work. Can you think of other numbers that would also work?

a. Are there numbers $a$ and $b$ with the property that $a^2 + b^2 = (a + b)^2$?
*Or*: Are there numbers $a$ and $b$ such that $a^2 + b^2 = (a + b)^2$?
*Or*: Do there exist any numbers $a$ and $b$ such that $a^2 + b^2 = (a + b)^2$?

b. Given any real number $r$, $r^2$ is nonnegative.
*Or*: For any real number $r$, $r^2 \geq 0$.
*Or*: For every real number $r$, $r^2 \geq 0$. ■

## *Some Important Kinds of Mathematical Statements*

Three of the most important kinds of sentences in mathematics are universal statements, conditional statements, and existential statements:

> A **universal statement** says that a certain property is true for all elements in a set. (For example: *All positive numbers are greater than zero.*)
>
> A **conditional statement** says that if one thing is true then some other thing also has to be true. (For example: *If 378 is divisible by 18, then 378 is divisible by 6.*)
>
> Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true. (For example: *There is a prime number that is even.*)

In later sections we will define each kind of statement carefully and discuss all of them in detail. The aim here is for you to realize that combinations of these statements can be expressed in a variety of different ways. One way uses ordinary, everyday language and another expresses the statement using one or more variables. The exercises are designed to help you start becoming comfortable in translating from one way to another.

## *Universal Conditional Statements*

Universal statements contain some variation of the words "for every" and conditional statements contain versions of the words "if-then." A **universal conditional statement** is a statement that is both universal and conditional. Here is an example:

> For every animal $a$, if $a$ is a dog, then $a$ is a mammal.

One of the most important facts about universal conditional statements is that they can be rewritten in ways that make them appear to be purely universal or purely conditional. For example, the previous statement can be rewritten in a way that makes its conditional nature explicit but its universal nature implicit:

> If $a$ is a dog, then $a$ is a mammal.
>
> *Or*: If an animal is a dog, then the animal is a mammal.

The statement can also be expressed so as to make its universal nature explicit and its conditional nature implicit:

> For every dog $a$, $a$ is a mammal.
> *Or*: All dogs are mammals.

The crucial point is that the ability to translate among various ways of expressing universal conditional statements is enormously useful for doing mathematics and many parts of computer science.

**Example 1.1.2**

### Rewriting a Universal Conditional Statement

Fill in the blanks to rewrite the following statement:

> For every real number $x$, if $x$ is nonzero then $x^2$ is positive.

**Note** If you introduce $x$ in the first part of the sentence, be sure to include it in the second part of the sentence.

a. If a real number is nonzero, then its square ______.
b. For every nonzero real number $x$, ______.
c. If $x$ ______, then ______.
d. The square of any nonzero real number is ______.
e. All nonzero real numbers have ______.

### Solution

a. is positive
b. $x^2$ is positive
c. is a nonzero real number; $x^2$ is positive
d. positive
e. positive squares (*or*: squares that are positive) ■

## *Universal Existential Statements*

**Note** For a number $b$ to be an additive inverse for a number $a$ means that $a + b = 0$.

A **universal existential statement** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something. For example:

> Every real number has an additive inverse.

In this statement the property "has an additive inverse" applies universally to all real numbers. "Has an additive inverse" asserts the existence of something—an additive inverse—for each real number. However, the nature of the additive inverse depends on the real number; different real numbers have different additive inverses. Knowing that an additive inverse is a real number, you can rewrite this statement in several ways, some less formal and some more formal:*

> All real numbers have additive inverses.
> *Or*: For every real number $r$, there is an additive inverse for $r$.
> *Or*: For every real number $r$, there is a real number $s$ such that $s$ is an additive inverse for $r$.

Introducing names for the variables simplifies references in further discussion. For instance, after the third version of the statement you might go on to write: When $r$ is positive, $s$ is negative, when $r$ is negative, $s$ is positive, and when $r$ is zero, $s$ is also zero.

One of the most important reasons for using variables in mathematics is that it gives you the ability to refer to quantities unambiguously throughout a lengthy mathematical argument, while not restricting you to consider only specific values for them.

### Example 1.1.3 Rewriting a Universal Existential Statement

Fill in the blanks to rewrite the following statement: Every pot has a lid.

a. All pots ________.

b. For every pot $P$, there is ________.

c. For every pot $P$, there is a lid $L$ such that ________.

**Solution**

a. have lids

b. a lid for $P$

c. $L$ is a lid for $P$ ■

## Existential Universal Statements

An **existential universal statement** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind. For example:

> There is a positive integer that is less than or equal to every positive integer.

This statement is true because the number one is a positive integer, and it satisfies the property of being less than or equal to every positive integer. We can rewrite the statement in several ways, some less formal and some more formal:

> Some positive integer is less than or equal to every positive integer.
> *Or*: There is a positive integer $m$ that is less than or equal to every positive integer.
> *Or*: There is a positive integer $m$ such that every positive integer is greater than or equal to $m$.
> *Or*: There is a positive integer $m$ with the property that for every positive integer $n$, $m \leq n$.

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*A conditional could be used to help express this statement, but we postpone the additional complexity to a later chapter.

### **Example 1.1.4** Rewriting an Existential Universal Statement

Fill in the blanks to rewrite the following statement in three different ways:

> There is a person in my class who is at least as old as every person in my class.

a. Some ________ is at least as old as ________.

b. There is a person $p$ in my class such that $p$ is ________.

c. There is a person $p$ in my class with the property that for every person $q$ in my class, $p$ is ________.

### Solution

a. person in my class; every person in my class

b. at least as old as every person in my class

c. at least as old as $q$ ■

Some of the most important mathematical concepts, such as the definition of limit of a sequence, can only be defined using phrases that are universal, existential, and conditional, and they require the use of all three phrases "for every," "there is," and "if-then." For example, if $a_1, a_2, a_3, \ldots$ is a sequence of real numbers, saying that

> the limit of $a_n$ as $n$ approaches infinity is $L$

means that

> **for every** positive real number $\varepsilon$, **there is** an integer $N$ such that
> **for every** integer $n$, **if** $n > N$ **then** $-\varepsilon < a_n - L < \varepsilon$.

## TEST YOURSELF

Answers to Test Yourself questions are located at the end of each section.

**1.** A universal statement asserts that a certain property is ________ for ________.

**2.** A conditional statement asserts that if one thing ________ then some other thing ________.

**3.** Given a property that may or may not be true, an existential statement asserts that ________ for which the property is true.

## EXERCISE SET 1.1

Appendix B contains either full or partial solutions to all exercises with blue numbers. When the solution is not complete, the exercise number has an "***H***" next to it. A "*" next to an exercise number signals that the exercise is more challenging than usual. Be careful not to get into the habit of turning to the solutions too quickly. Make every effort to work exercises on your own before checking your answers. See the Preface for additional sources of assistance and further study.

In each of 1–6, fill in the blanks using a variable or variables to rewrite the given statement.

**1.** Is there a real number whose square is $-1$?
  **a.** Is there a real number $x$ such that ______?
  **b.** Does there exist ______ such that $x^2 = -1$?

**2.** Is there an integer that has a remainder of 2 when it is divided by 5 and a remainder of 3 when it is divided by 6?
  **a.** Is there an integer $n$ such that $n$ has ______?
  **b.** Does there exist ______ such that if $n$ is divided by 5 the remainder is 2 and if ______?

*Note: There are integers with this property. Can you think of one?*

**3.** Given any two distinct real numbers, there is a real number in between them.

**a.** Given any two distinct real numbers *a* and *b*, there is a real number *c* such that *c* is ______.

**b.** For any two ______, ______ such that *c* is between *a* and *b*.

**4.** Given any real number, there is a real number that is greater.

**a.** Given any real number *r*, there is ______ *s* such that *s* is ______.

**b.** For any ______, ______ such that $s > r$.

**5.** The reciprocal of any positive real number is positive.

**a.** Given any positive real number *r*, the reciprocal of ______.

**b.** For any real number *r*, if *r* is ______, then ______.

**c.** If a real number *r* ______, then ______.

**6.** The cube root of any negative real number is negative.

**a.** Given any negative real number *s*, the cube root of ______.

**b.** For any real number *s*, if *s* is ______, then ______.

**c.** If a real number *s* ______, then ______.

**7.** Rewrite the following statements less formally, without using variables. Determine, as best as you can, whether the statements are true or false.

**a.** There are real numbers *u* and *v* with the property that $u + v < u - v$.

**b.** There is a real number *x* such that $x^2 < x$.

**c.** For every positive integer *n*, $n^2 \geq n$.

**d.** For all real numbers *a* and *b*, $|a + b| \leq |a| + |b|$.

**In each of 8–13, fill in the blanks to rewrite the given statement.**

**8.** For every object *J*, if *J* is a square then *J* has four sides.

**a.** All squares ______.

**b.** Every square ______.

**c.** If an object is a square, then it ______.

**d.** If *J* ______, then *J* ______.

**e.** For every square *J*, ______.

**9.** For every equation *E*, if *E* is quadratic then *E* has at most two real solutions.

**a.** All quadratic equations ______.

**b.** Every quadratic equation ______.

**c.** If an equation is quadratic, then it ______.

**d.** If *E* ______, then *E* ______.

**e.** For every quadratic equation *E*, ______.

**10.** Every nonzero real number has a reciprocal.

**a.** All nonzero real numbers ______.

**b.** For every nonzero real number *r*, there is ______ for *r*.

**c.** For every nonzero real number *r*, there is a real number *s* such that ______.

**11.** Every positive number has a positive square root.

**a.** All positive numbers ______.

**b.** For every positive number *e*, there is ______ for *e*.

**c.** For every positive number *e*, there is a positive number *r* such that ______.

**12.** There is a real number whose product with every number leaves the number unchanged.

**a.** Some ______ has the property that its ______.

**b.** There is a real number *r* such that the product of *r* ______.

**c.** There is a real number *r* with the property that for every real number *s*, ______.

**13.** There is a real number whose product with every real number equals zero.

**a.** Some ______ has the property that its ______.

**b.** There is a real number *a* such that the product of *a* ______.

**c.** There is a real number *a* with the property that for every real number *b*, ______.

## ANSWERS FOR TEST YOURSELF

**1.** true; all elements of a set **2.** is true; also has to be true **3.** there is at least one thing

## 1.2 The Language of Sets

> *. . . when we attempt to express in mathematical symbols a condition proposed in words. First, we must understand thoroughly the condition. Second, we must be familiar with the forms of mathematical expression.* —George Polyá (1887–1985)

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor (1845–1918). For most mathematical purposes we can think of a set intuitively, as

Cantor did, simply as a collection of elements. For instance, if $C$ is the set of all countries that are currently in the United Nations, then the United States is an element of $C$, and if $I$ is the set of all integers from 1 to 100, then the number 57 is an element of $I$.

### Set-Roster Notation

If $S$ is a set, the notation $\boldsymbol{x \in S}$ means that $x$ is an element of $S$. The notation $\boldsymbol{x \notin S}$ means that $x$ is not an element of $S$. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, **{1, 2, 3}** denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write **{1, 2, 3, . . . , 100}** to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write **{1, 2, 3, . . . }** to refer to the set of all positive integers. (The symbol . . . is called an **ellipsis** and is read “and so forth.”)

The **axiom of extension** says that a set is completely determined by what its elements are—not the order in which they might be listed or the fact that some elements might be listed more than once.

**Example 1.2.1**

## Using the Set-Roster Notation

a. Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of $A$, $B$, and $C$? How are $A$, $B$, and $C$ related?

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1, \{1\}\}$?

d. For each nonnegative integer $n$, let $U_n = \{n, -n\}$. Find $U_1$, $U_2$, and $U_0$.

### Solution

a. $A$, $B$, and $C$ have exactly the same three elements: 1, 2, and 3. Therefore, $A$, $B$, and $C$ are simply different ways to represent the same set.

b. $\{0\} \neq 0$ because $\{0\}$ is a set with one element, namely 0, whereas 0 is just the symbol that represents the number zero.

c. The set $\{1, \{1\}\}$ has two elements: 1 and the set whose only element is 1.

d. $U_1 = \{1, -1\}$, $U_2 = \{2, -2\}$, $U_0 = \{0, -0\} = \{0, 0\} = \{0\}$. ■

Certain sets of numbers are so frequently referred to that they are given special symbolic names. These are summarized in the following table.

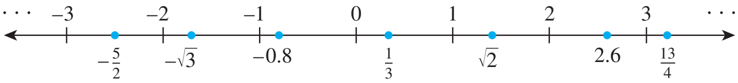
**Note** The **Z** is the first letter of the German word for integers, *Zahlen*. It stands for the *set* of all integers and should not be used as a shorthand for the word *integer*.

When the Symbols **R**, **Q**, and **Z** are handwritten, they appear as $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$.

| Symbol | Set |
|---|---|
| **R** | the set of all real numbers |
| **Z** | the set of all integers |
| **Q** | the set of all rational numbers, or quotients of integers |

Addition of a superscript + or − or the letters *nonneg* indicates that only the positive or negative or nonnegative elements of the set, respectively, are to be included. Thus $\mathbf{R}^+$ denotes the set of positive real numbers, and $\mathbf{Z}^{nonneg}$ refers to the set of nonnegative integers: 0, 1, 2, 3, 4, and so forth. Some authors refer to the set of nonnegative integers as the set of **natural numbers** and denote it as **N**. Other authors call only the positive integers natural numbers. To prevent confusion, we simply avoid using the phrase *natural numbers* in this book.

The set of real numbers is usually pictured as the set of all points on a line, as shown below. The number 0 corresponds to a middle point, called the *origin*. A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin. Each point to the left of the origin corresponds to a negative real number, which is denoted by computing its distance from the origin and putting a minus sign in front of the resulting number. The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0. *Note that 0 is neither positive nor negative*. Labels are given for a few real numbers corresponding to points on the line shown below.



The real number line is called *continuous* because it is imagined to have no holes. The set of integers corresponds to a collection of points located at fixed intervals along the real number line. Thus every integer is a real number, and because the integers are all separated from each other, the set of integers is called *discrete*. The name *discrete mathematics* comes from the distinction between continuous and discrete mathematical objects.

Another way to specify a set uses what is called the *set-builder notation*.

**Note** We read the left-hand brace as "the set of all" and the vertical line as "such that." In all other mathematical contexts, however, we do not use a vertical line to denote the words "such that"; we abbreviate "such that" as "s. t." or "s. th." or "·∋·."

### Set-Builder Notation

Let $S$ denote a set and let $P(x)$ be a property that elements of $S$ may or may not satisfy. We may define a new set to be **the set of all elements $x$ in $S$ such that $P(x)$ is true**. We denote this set as follows:

$$\underset{\substack{\nearrow \\ \text{the set of all}}}{\{}x \in S \underset{\substack{\nwarrow \\ \text{such that}}}{\mid} P(x)\}$$

Occasionally we will write $\{x \mid P(x)\}$ without being specific about where the element $x$ comes from. It turns out that unrestricted use of this notation can lead to genuine contradictions in set theory. We will discuss one of these in Section 6.4 and will be careful to use this notation purely as a convenience in cases where the set $S$ could be specified if necessary.

## Example 1.2.2 Using the Set-Builder Notation

Given that **R** denotes the set of all real numbers, **Z** the set of all integers, and $\mathbf{Z}^+$ the set of all positive integers, describe each of the following sets.

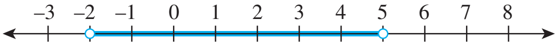
a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$

b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$

c. $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\}$

### Solution

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between $-2$ and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between $-2$ and 5. It is equal to the set $\{-1, 0, 1, 2, 3, 4\}$.

c. Since all the integers in $\mathbf{Z}^+$ are positive, $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}$. ■

## Subsets

A basic relation between sets is that of subset.

### Definition

If $A$ and $B$ are sets, then $A$ is called a **subset** of $B$, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of $A$ is also an element of $B$.

Symbolically:

$$A \subseteq B \quad \text{means that} \quad \text{for every element } x, \text{ if } x \in A \text{ then } x \in B.$$

The phrases $A$ *is contained in* $B$ and $B$ *contains* $A$ are alternative ways of saying that $A$ is a subset of $B$.

It follows from the definition of subset that for a set $A$ not to be a subset of a set $B$ means that there is at least one element of $A$ that is not an element of $B$. Symbolically:

$$A \not\subseteq B \quad \text{means that} \quad \text{there is at least one element } x \text{ such that } x \in A \text{ and } x \notin B.$$

### Definition

Let $A$ and $B$ be sets. $A$ is a **proper subset** of $B$ if, and only if, every element of $A$ is in $B$ but there is at least one element of $B$ that is not in $A$.

**Example 1.2.3**

### Subsets

Let $A = \mathbf{Z}^+$, $B = \{n \in \mathbf{Z} \mid 0 \leq n \leq 100\}$, and $C = \{100, 200, 300, 400, 500\}$. Evaluate the truth and falsity of each of the following statements.

a. $B \subseteq A$

b. $C$ is a proper subset of $A$

c. $C$ and $B$ have at least one element in common

d. $C \subseteq B$

e. $C \subseteq C$

### Solution

a. False. Zero is not a positive integer. Thus zero is in $B$ but zero is not in $A$, and so $B \not\subseteq A$.

b. True. Each element in $C$ is a positive integer and, hence, is in $A$, but there are elements in $A$ that are not in $C$. For instance, 1 is in $A$ and not in $C$.

c. True. For example, 100 is in both $C$ and $B$.

d. False. For example, 200 is in $C$ but not in $B$.

e. True. Every element in $C$ is in $C$. In general, the definition of subset implies that all sets are subsets of themselves. ■

**Example 1.2.4**

### Distinction between $\in$ and $\subseteq$

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ b. $\{2\} \in \{1, 2, 3\}$ c. $2 \subseteq \{1, 2, 3\}$

d. $\{2\} \subseteq \{1, 2, 3\}$ e. $\{2\} \subseteq \{\{1\}, \{2\}\}$ f. $\{2\} \in \{\{1\}, \{2\}\}$

**Solution** Only (a), (d), and (f) are true.

For (b) to be true, the set $\{1, 2, 3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1, 2, 3\}$ are 1, 2, and 3, and 2 is not equal to $\{2\}$. Hence (b) is false.

For (c) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of $\{1, 2, 3\}$. This is not the case, so (c) is false.

For (e) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are $\{1\}$ and $\{2\}$. But 2 is not equal to either $\{1\}$ or $\{2\}$, and so (e) is false. ■

## Cartesian Products

ArchivePL/Alamy Stock Photo

*Kazimierz Kuratowski (1896–1980)*

With the introduction of Georg Cantor's set theory in the late nineteenth century, it began to seem possible to put mathematics on a firm logical foundation by developing all of its various branches from set theory and logic alone. A major stumbling block was how to use sets to define an ordered pair because the definition of a set is unaffected by the order in which its elements are listed. For example, $\{a, b\}$ and $\{b, a\}$ represent the same set, whereas in an ordered pair we want to be able to indicate which element comes first.

In 1914 crucial breakthroughs were made by Norbert Wiener (1894–1964), a young American who had recently received his Ph.D. from Harvard, and the German mathematician Felix Hausdorff (1868–1942). Both gave definitions showing that an ordered pair can be defined as a certain type of set, but both definitions were somewhat awkward. Finally, in 1921, the Polish mathematician Kazimierz Kuratowski (1896–1980) published the following definition, which has since become standard. It says that an ordered pair is a set of the form

$$\{\{a\}, \{a, b\}\}.$$

This set has elements, $\{a\}$ and $\{a, b\}$. If $a \neq b$, then the two sets are distinct and $a$ is in both sets whereas $b$ is not. This allows us to distinguish between $a$ and $b$ and say that $a$ is the first element of the ordered pair and $b$ is the second element of the pair. If $a = b$, then we can simply say that $a$ is both the first and the second element of the pair. In this case the set that defines the ordered pair becomes $\{\{a\}, \{a, a\}\}$, which equals $\{\{a\}\}$.

However, it was only long after ordered pairs had been used extensively in mathematics that mathematicians realized that it was possible to define them entirely in terms of sets, and, in any case, the set notation would be cumbersome to use on a regular basis. The usual notation for ordered pairs refers to $\{\{a\}, \{a, b\}\}$ more simply as $(a, b)$.

**Notation**

Given elements $a$ and $b$, the symbol $(a, b)$ denotes the **ordered pair** consisting of $a$ and $b$ together with the specification that $a$ is the first element of the pair and $b$ is the second element. Two ordered pairs $(a, b)$ and $(c, d)$ are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

**Example 1.2.5**

### Ordered Pairs

a. Is (1, 2) = (2, 1)?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of (1, 1)?

### Solution

a. No. By definition of equality of ordered pairs,

$$(1, 2) = (2, 1) \text{ if, and only if, } 1 = 2 \text{ and } 2 = 1.$$

But 1 ≠ 2, and so the ordered pairs are not equal.

b. Yes. By definition of equality of ordered pairs,

$$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right) \text{ if, and only if, } 3 = \sqrt{9} \text{ and } \frac{5}{10} = \frac{1}{2}.$$

Because these equations are both true, the ordered pairs are equal.

c. In the ordered pair (1, 1), the first and the second elements are both 1. ■

The notation for an *ordered n-tuple* generalizes the notation for an ordered pair to a set with any finite number of elements. It also takes both order and multiplicity into account.

**Definition**

Let $n$ be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered $n$-tuple, $(x_1, x_2, \ldots, x_n)$**, consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first $x_1$, then $x_2$, and so forth up to $x_n$. An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered $n$-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots,$ and $x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

**Example 1.2.6**

### Ordered *n*-tuples

a. Is (1, 2, 3, 4) = (1, 2, 4, 3)?

b. Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?

### Solution

a. No. By definition of equality of ordered 4-tuples,

$$(1, 2, 3, 4) = (1, 2, 4, 3) \Leftrightarrow 1 = 1, 2 = 2, 3 = 4, \text{ and } 4 = 3$$

But 3 ≠ 4, and so the ordered 4-tuples are not equal.

b. Yes. By definition of equality of ordered triples,

$$\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \frac{1}{2} = \frac{3}{6}.$$

Because these equations are all true, the two ordered triples are equal. ■

**Definition**

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $\mathbf{A_1 \times A_2 \times \cdots \times A_n}$, is the set of all ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of $A_1$ and $A_2$.

## Example 1.2.7 Cartesian Products

Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$.

a. Find $A \times B$.

b. Find $B \times A$.

c. Find $A \times A$.

d. How many elements are in $A \times B$, $B \times A$, and $A \times A$?

e. Find $(A \times B) \times C$

f. Find $A \times B \times C$

g. Let $\mathbf{R}$ denote the set of all real numbers. Describe $\mathbf{R} \times \mathbf{R}$.

### Solution

a. $A \times B = \{(x, 1), (y, 1), (x, 2), (y, 2), (x, 3), (y, 3)\}$

b. $B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

c. $A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$

**Note** This is why it makes sense to call a Cartesian product a product!

d. $A \times B$ has 6 elements. Note that this is the number of elements in $A$ times the number of elements in $B$. $B \times A$ has 6 elements, the number of elements in $B$ times the number of elements in $A$. $A \times A$ has 4 elements, the number of elements in $A$ times the number of elements in $A$.

e. The Cartesian product of $A$ and $B$ is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A \times B) \times C$.

$$\begin{aligned} (A \times B) \times C &= \{(u, v) \mid u \in A \times B \text{ and } v \in C\} && \text{by definition of Cartesian product} \\ &= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a), \\ &\quad ((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b), \\ &\quad ((y, 1), b), ((y, 2), b), ((y, 3), b)\} \end{aligned}$$

f. The Cartesian product $A \times B \times C$ is superficially similar to but is not quite the same mathematical object as $(A \times B) \times C$. $(A \times B) \times C$ is a set of ordered pairs of which one element is itself an ordered pair, whereas $A \times B \times C$ is a set of ordered triples. By definition of Cartesian product,

$$\begin{aligned} A \times B \times C &= \{(u, v, w) \,|\, u \in A, v \in B, \text{ and } w \in C\} \\ &= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a), (y, 3, a), (x, 1, b), \\ &\quad\ (x, 2, b), (x, 3, b), (y, 1, b), (y, 2, b), (y, 3, b)\}. \end{aligned}$$

g. $\mathbf{R} \times \mathbf{R}$ is the set of all ordered pairs $(x, y)$ where both $x$ and $y$ are real numbers. If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in $\mathbf{R} \times \mathbf{R}$ corresponds to a unique point in the plane, with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point. The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in Figure 1.2.1.

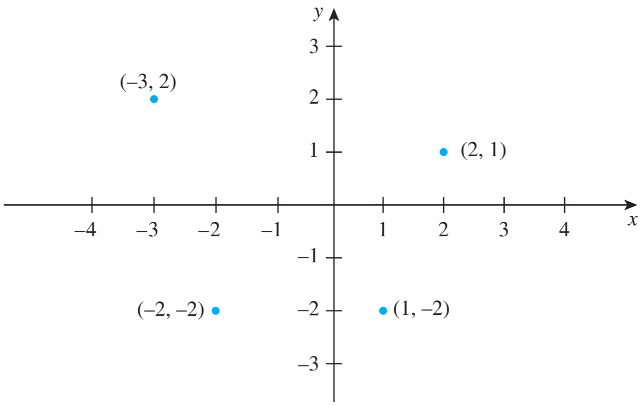


**FIGURE 1.2.1** A Cartesian Plane ■

Another notation, which is important in both mathematics and computer science, denotes objects called *strings*.*

**Definition**

Let $n$ be a positive integer. Given a finite set $A$, a **string of length $n$ over $A$** is an ordered $n$-tuple of elements of $A$ written without parentheses or commas. The elements of $A$ are called the **characters** of the string. The **null string** over $A$ is defined to be the "string" with no characters. It is often denoted $\lambda$ and is said to have length 0. If $A = \{0, 1\}$, then a string over $A$ is called a **bit string**.

**Example 1.2.8**

### Strings

Let $A = \{a, b\}$. List all the strings of length 3 over $A$ with at least two characters that are the same.

**Solution**

$$aab, aba, baa, aaa, bba, bab, abb, bbb$$

In computer programming it is important to distinguish among different kinds of data structures and to respect the notations that are used for them. Similarly in mathematics, it is important to distinguish among, say, $\{a, b, c\}$, $\{\{a, b\}, c\}$, $(a, b, c)$, $(a, (b, c))$, $abc$, and so forth, because these are all significantly different objects. ■

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*A more formal definition of string, using recursion, is given in Section 5.9.

## TEST YOURSELF

**1.** When the elements of a set are given using the set-roster notation, the order in which they are listed ________.

**2.** The symbol **R** denotes ________.

**3.** The symbol **Z** denotes ________.

**4.** The symbol **Q** denotes ________.

**5.** The notation $\{x \mid P(x)\}$ is read ________.

**6.** For a set $A$ to be a subset of a set $B$ means that ________.

**7.** Given sets $A$ and $B$, the Cartesian product $A \times B$ is ________.

**8.** Given sets $A$, $B$, and $C$, the Cartesian product $A \times B \times C$ is ________.

**9.** A string of length $n$ over a set $S$ is an ordered $n$-tuple of elements of $S$, written without ________ or ________.

## EXERCISE SET 1.2

**1.** Which of the following sets are equal?

$$A = \{a, b, c, d\} \qquad B = \{d, e, a, c\}$$
$$C = \{d, b, a, c\} \qquad D = \{a, a, d, e, c, e\}$$

**2.** Write in words how to read each of the following out loud.
- **a.** $\{x \in \mathbf{R}^+ \mid 0 < x < 1\}$
- **b.** $\{x \in \mathbf{R} \mid x \le 0 \text{ or } x \ge 1\}$
- **c.** $\{n \in \mathbf{Z} \mid n \text{ is a factor of } 6\}$
- **d.** $\{n \in \mathbf{Z}^+ \mid n \text{ is a factor of } 6\}$

**3.** **a.** Is $4 = \{4\}$?
- **b.** How many elements are in the set $\{3, 4, 3, 5\}$?
- **c.** How many elements are in the set $\{1, \{1\}, \{1, \{1\}\}\}$?

**4.** **a.** Is $2 \in \{2\}$?
- **b.** How many elements are in the set $\{2, 2, 2, 2\}$?
- **c.** How many elements are in the set $\{0, \{0\}\}$?
- **d.** Is $\{0\} \in \{\{0\}, \{1\}\}$?
- **e.** Is $0 \in \{\{0\}, \{1\}\}$?

***H* 5.** Which of the following sets are equal?

$$A = \{0, 1, 2\}$$
$$B = \{x \in \mathbf{R} \mid -1 \le x < 3\}$$
$$C = \{x \in \mathbf{R} \mid -1 < x < 3\}$$
$$D = \{x \in \mathbf{Z} \mid -1 < x < 3\}$$
$$E = \{x \in \mathbf{Z}^+ \mid -1 < x < 3\}$$

***H* 6.** For each integer $n$, let $T_n = \{n, n^2\}$. How many elements are in each of $T_2$, $T_{-3}$, $T_1$, and $T_0$? Justify your answers.

**7.** Use the set-roster notation to indicate the elements in each of the following sets.
- **a.** $S = \{n \in \mathbf{Z} \mid n = (-1)^k, \text{ for some integer } k\}$.
- **b.** $T = \{m \in \mathbf{Z} \mid m = 1 + (-1)^i, \text{ for some integer } i\}$.
- **c.** $U = \{r \in \mathbf{Z} \mid 2 \le r \le -2\}$
- **d.** $V = \{s \in \mathbf{Z} \mid s > 2 \text{ or } s < 3\}$
- **e.** $W = \{t \in \mathbf{Z} \mid 1 < t < -3\}$
- **f.** $X = \{u \in \mathbf{Z} \mid u \le 4 \text{ or } u \ge 1\}$

**8.** Let $A = \{c, d, f, g\}$, $B = \{f, j\}$, and $C = \{d, g\}$. Answer each of the following questions. Give reasons for your answers.
- **a.** Is $B \subseteq A$?
- **b.** Is $C \subseteq A$?
- **c.** Is $C \subseteq C$?
- **d.** Is $C$ a proper subset of $A$?

**9.** **a.** Is $3 \in \{1, 2, 3\}$?
- **b.** Is $1 \subseteq \{1\}$?
- **c.** Is $\{2\} \in \{1, 2\}$?
- **d.** Is $\{3\} \in \{1, \{2\}, \{3\}\}$?
- **e.** Is $1 \in \{1\}$?
- **f.** Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$?
- **g.** Is $\{1\} \subseteq \{1, 2\}$?
- **h.** Is $1 \in \{\{1\}, 2\}$?
- **i.** Is $\{1\} \subseteq \{1, \{2\}\}$?
- **j.** Is $\{1\} \subseteq \{1\}$?

**10.** **a.** Is $((-2)^2, -2^2) = (-2^2, (-2)^2)$?
- **b.** Is $(5, -5) = (-5, 5)$?
- **c.** Is $(8 - 9, \sqrt[3]{-1}) = (-1, -1)$?
- **d.** Is $\left(\frac{-2}{-4}, (-2)^3\right) = \left(\frac{3}{6}, -8\right)$?

**11.** Let $A = \{w, x, y, z\}$ and $B = \{a, b\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set.
- **a.** $A \times B$
- **b.** $B \times A$
- **c.** $A \times A$
- **d.** $B \times B$

**12.** Let $S = \{2, 4, 6\}$ and $T = \{1, 3, 5\}$. Use the set-roster notation to write each of the following sets, and indicate the number of elements that are in each set.

**a.** $S \times T$
**b.** $T \times S$
**c.** $S \times S$
**d.** $T \times T$

**13.** Let $A = \{1, 2, 3\}$, $B = \{u\}$, and $C = \{m, n\}$. Find each of the following sets.

**a.** $A \times (B \times C)$
**b.** $(A \times B) \times C$
**c.** $A \times B \times C$

**14.** Let $R = \{a\}$, $S = \{x, y\}$, and $T = \{p, q, r\}$. Find each of the following sets.

**a.** $R \times (S \times T)$
**b.** $(R \times S) \times T$
**c.** $R \times S \times T$

**15.** Let $S = \{0, 1\}$. List all the strings of length 4 over $S$ that contain three or more 0's.

**16.** Let $T = \{x, y\}$. List all the strings of length 5 over $T$ that have exactly one $y$.

## ANSWERS FOR TEST YOURSELF

**1.** does not matter **2.** the set of all real numbers **3.** the set of all integers **4.** the set of all rational numbers **5.** the set of all $x$ such that $P(x)$ **6.** every element in $A$ is an element in $B$ **7.** the set of all ordered pairs $(a, b)$ where $a$ is in $A$ and $b$ is in $B$ **8.** the set of ordered triples of the form $(a, b, c)$ where $a \in A$, $b \in B$, and $c \in C$ **9.** parentheses; commas

# 1.3 The Language of Relations and Functions

*Mathematics is a language.* —Josiah Willard Gibbs (1839–1903)

There are many kinds of relationships in the world. For instance, we say that two people are related by blood if they share a common ancestor and that they are related by marriage if one shares a common ancestor with the spouse of the other. We also speak of the relationship between student and teacher, between people who work for the same employer, and between people who share a common ethnic background.

Similarly, the objects of mathematics may be related in various ways. A set $A$ may be said to be related to a set $B$ if $A$ is a subset of $B$, or if $A$ is not a subset of $B$, or if $A$ and $B$ have at least one element in common. A number $x$ may be said to be related to a number $y$ if $x < y$, or if $x$ is a factor of $y$, or if $x^2 + y^2 = 1$. Two identifiers in a computer program may be said to be related if they have the same first eight characters, or if the same memory location is used to store their values when the program is executed. And the list could go on!

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$ and let us say that an element $x$ in $A$ is related to an element $y$ in $B$ if, and only if, $x$ is less than $y$. Let us use the notation $x\ R\ y$ as a shorthand for the sentence "$x$ is related to $y$." Then

$$\begin{array}{lll}
0\ R\ 1 & \text{since} & 0 < 1, \\
0\ R\ 2 & \text{since} & 0 < 2, \\
0\ R\ 3 & \text{since} & 0 < 3, \\
1\ R\ 2 & \text{since} & 1 < 2, \\
1\ R\ 3 & \text{since} & 1 < 3, \quad \text{and} \\
2\ R\ 3 & \text{since} & 2 < 3.
\end{array}$$

On the other hand, if the notation $x \not R y$ represents the sentence "$x$ is not related to $y$," then

$$\begin{aligned} 1 \not R 1 &\quad \text{since} \quad 1 \not< 1, \\ 2 \not R 1 &\quad \text{since} \quad 2 \not< 1, \quad \text{and} \\ 2 \not R 2 &\quad \text{since} \quad 2 \not< 2. \end{aligned}$$

Recall that the Cartesian product of $A$ and $B$, $A \times B$, consists of all ordered pairs whose first element is in $A$ and whose second element is in $B$:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

In this case,

$$A \times B = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

The elements of some ordered pairs in $A \times B$ are related, whereas the elements of other ordered pairs are not. Consider the set of all ordered pairs in $A \times B$ whose elements are related

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

Observe that knowing which ordered pairs lie in this set is equivalent to knowing which elements are related to which. The relation itself can therefore be thought of as the totality of ordered pairs whose elements are related by the given condition. The formal mathematical definition of relation, based on this idea, was introduced by the American mathematician and logician C. S. Peirce in the nineteenth century.

### Definition

Let $A$ and $B$ be sets. A **relation $R$ from $A$ to $B$** is a subset of $A \times B$. Given an ordered pair $(x, y)$ in $A \times B$, **$x$ is related to $y$ by $R$**, written $x \, R \, y$, if, and only if, $(x, y)$ is in $R$. The set $A$ is called the **domain** of $R$ and the set $B$ is called its **co-domain**.

The notation for a relation $R$ may be written symbolically as follows:

$$\boldsymbol{x \, R \, y} \quad \text{means that} \quad \boldsymbol{(x, y) \in R.}$$

The notation $x \not R y$ means that $x$ is not related to $y$ by $R$:

$$\boldsymbol{x \not R y} \quad \text{means that} \quad \boldsymbol{(x, y) \notin R.}$$

## Example 1.3.1 A Relation as a Subset

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation $R$ from $A$ to $B$ as follows: Given any $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad \frac{x - y}{2} \text{ is an integer.}$$

a. State explicitly which ordered pairs are in $A \times B$ and which are in $R$.

b. Is $1 \, R \, 3$? Is $2 \, R \, 3$? Is $2 \, R \, 2$?

c. What are the domain and co-domain of $R$?

### Solution

a. $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$. To determine explicitly the composition of $R$, examine each ordered pair in $A \times B$ to see whether its elements satisfy the defining condition for $R$.

$$(1, 1) \in R \text{ because } \frac{1-1}{2} = \frac{0}{2} = 0, \text{ which is an integer.}$$

$$(1, 2) \notin R \text{ because } \frac{1-2}{2} = \frac{-1}{2}, \text{ which is not an integer.}$$

$$(1, 3) \in R \text{ because } \frac{1-3}{2} = \frac{-2}{2} = -1, \text{ which is an integer.}$$

$$(2, 1) \notin R \text{ because } \frac{2-1}{2} = \frac{1}{2}, \text{ which is not an integer.}$$

$$(2, 2) \in R \text{ because } \frac{2-2}{2} = \frac{0}{2} = 0, \text{ which is an integer.}$$

$$(2, 3) \notin R \text{ because } \frac{2-3}{2} = \frac{-1}{2}, \text{ which is not an integer.}$$

Thus

$$R = \{(1, 1), (1, 3), (2, 2)\}$$

b. Yes, $1 \, R \, 3$ because $(1, 3) \in R$.
No, $2 \not{R}\, 3$ because $(2, 3) \notin R$.
Yes, $2 \, R \, 2$ because $(2, 2) \in R$.

c. The domain of $R$ is $\{1, 2\}$ and the co-domain is $\{1, 2, 3\}$. ■

**Example 1.3.2**

### The Circle Relation

Define a relation $C$ from $\mathbf{R}$ to $\mathbf{R}$ as follows: For any $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in C \quad \text{means that} \quad x^2 + y^2 = 1.$$

a. Is $(1, 0) \in C$? Is $(0, 0) \in C$? Is $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$? Is $-2 \; C \; 0$? Is $0 \; C \; (-1)$? Is $1 \; C \; 1$?

b. What are the domain and co-domain of $C$?

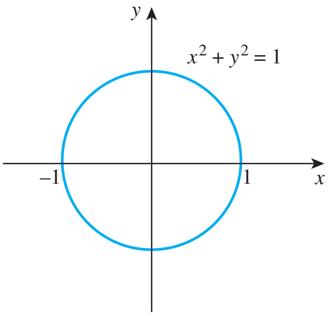
c. Draw a graph for $C$ by plotting the points of $C$ in the Cartesian plane.

### Solution

a. Yes, $(1, 0) \in C$ because $1^2 + 0^2 = 1$.
No, $(0, 0) \notin C$ because $0^2 + 0^2 = 0 \neq 1$.
Yes, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in C$ because $\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$.
No, $-2 \not{C}\, 0$ because $(-2)^2 + 0^2 = 4 \neq 1$.
Yes, $0 \; C \; (-1)$ because $0^2 + (-1)^2 = 1$.
No, $1 \not{C}\, 1$ because $1^2 + 1^2 = 2 \neq 1$.

b. The domain and co-domain of $C$ are both $\mathbf{R}$, the set of all real numbers.

c.



■

## *Arrow Diagram of a Relation*

Suppose $R$ is a relation from a set $A$ to a set $B$. The **arrow diagram for $R$** is obtained as follows:

1. Represent the elements of $A$ as points in one region and the elements of $B$ as points in another region.
2. For each $x$ in $A$ and $y$ in $B$, draw an arrow from $x$ to $y$ if, and only if, $x$ is related to $y$ by $R$. Symbolically:

$$\begin{aligned} &\textbf{Draw an arrow from } \boldsymbol{x} \textbf{ to } \boldsymbol{y} \\ &\quad \textbf{if, and only if,} \qquad \boldsymbol{x\ R\ y} \\ &\quad \textbf{if, and only if,} \qquad \boldsymbol{(x, y) \in R}. \end{aligned}$$

**Example 1.3.3**

### Arrow Diagrams of Relations

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations $S$ and $T$ from $A$ to $B$ as follows: For every $(x, y) \in A \times B$,

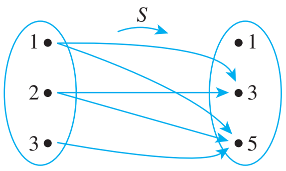
$$(x, y) \in S \quad \text{means that} \quad x < y$$

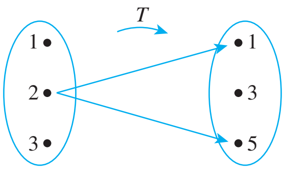
$S$ is a "less than" relation.

$$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for $S$ and $T$.

### Solution





These example relations illustrate that it is possible to have several arrows coming out of the same element of $A$ pointing in different directions. Also, it is quite possible to have an element of $A$ that does not have an arrow coming out of it. ■

## *Functions*

In Section 1.2 we showed that ordered pairs can be defined in terms of sets and we defined Cartesian products in terms of ordered pairs. In this section we introduced relations as subsets of Cartesian products. Thus we can now define functions in a way that depends only on the concept of set. Although this definition is not obviously related to the way we usually work with functions in mathematics, it is satisfying from a theoretical point of view, and computer scientists like it because it is particularly well suited for operating with functions on a computer.

### Definition

A **function $F$ from a set $A$ to a set $B$** is a relation with domain $A$ and co-domain $B$ that satisfies the following two properties:

1. For every element $x$ in $A$, there is an element $y$ in $B$ such that $(x, y) \in F$.
2. For all elements $x$ in $A$ and $y$ and $z$ in $B$,

$$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

Properties (1) and (2) can be stated less formally as follows: A relation $F$ from $A$ to $B$ is a function if, and only if:

1. Every element of $A$ is the first element of an ordered pair of $F$.
2. No two distinct ordered pairs in $F$ have the same first element.

In most mathematical situations we think of a function as sending elements from one set, the domain, to elements of another set, the co-domain. Because of the definition of function, each element in the domain corresponds to one and only one element of the co-domain.

More precisely, if $F$ is a function from a set $A$ to a set $B$, then given any element $x$ in $A$, property (1) from the function definition guarantees that there is at least one element of $B$ that is related to $x$ by $F$ and property (2) guarantees that there is at most one such element. This makes it possible to give the element that corresponds to $x$ a special name.

### Function Notation

If $A$ and $B$ are sets and $F$ is a function from $A$ to $B$, then given any element $x$ in $A$, the unique element in $B$ that is related to $x$ by $F$ is denoted $\boldsymbol{F(x)}$, which is read "$\boldsymbol{F}$ **of** $\boldsymbol{x}$."

**Example 1.3.4**

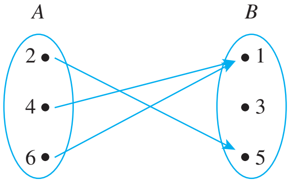
### Functions and Relations on Finite Sets

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations $R$, $S$, and $T$ defined below are functions from $A$ to $B$?

a. $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$

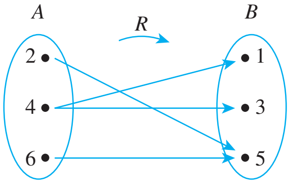
b. For every $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.

c. $T$ is defined by the arrow diagram

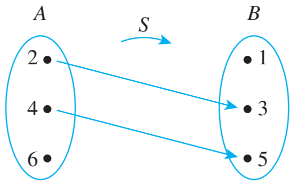


### Solution

a. $R$ is not a function because it does not satisfy property (2). The ordered pairs (4, 1) and (4, 3) have the same first element but different second elements. You can see this graphically if you draw the arrow diagram for $R$. There are two arrows coming out of 4: One points to 1 and the other points to 3.



b. $S$ is not a function because it does not satisfy property (1). It is not true that every element of $A$ is the first element of an ordered pair in $S$. For example, $6 \in A$ but there is no $y$ in $B$ such that $y = 6 + 1 = 7$. You can also see this graphically by drawing the arrow diagram for $S$.



**Note** In part (c), $T(4) = T(6)$. This illustrates the fact that although no element of the domain of a function can be related to more than one element of the co-domain, several elements in the domain can be related to the same element in the co-domain.

c. $T$ is a function: Each element in $\{2, 4, 6\}$ is related to some element in $\{1, 3, 5\}$, and no element in $\{2, 4, 6\}$ is related to more than one element in $\{1, 3, 5\}$. When these properties are stated in terms of the arrow diagram, they become (1) there is an arrow coming out of each element of the domain, and (2) no element of the domain has more than one arrow coming out of it. So you can write $T(2) = 5$, $T(4) = 1$, and $T(6) = 1$. ■

### Example 1.3.5 Functions and Relations on Sets of Strings

Let $A = \{a, b\}$ and let $S$ be the set of all strings over $A$.

a. Define a relation $L$ from $S$ to $\mathbf{Z}^{nonneg}$ as follows: For every string $s$ in $S$ and for every nonnegative integer $n$,

$$(s, n) \in L \text{ means that the length of } s \text{ is } n.$$

Observe that $L$ is a function because every string in $S$ has one and only one length. Find $L(abaaba)$ and $L(bbb)$.

b. Define a relation $C$ from $S$ to $S$ as follows: For all strings $s$ and $t$ in $S$,

$$(s, t) \in C \text{ means that } t = as,$$

where $as$ is the string obtained by appending $a$ on the left of the characters in $s$. ($C$ is called **concatenation** by $a$ on the left.) Observe that $C$ is a function because every string in $S$ consists entirely of $a$'s and $b$'s and adding an additional $a$ on the left creates a new strong that also consists of $a$'s and $b$'s and thus is also in $S$. Find $C(abaaba)$ and $C(bbb)$.

#### Solution

a. $L(abaaba) = 6$ and $L(bbb) = 3$

b. $C(abaaba) = aabaaba$ and $C(bbb) = abbb$ ■

## Function Machines

Another useful way to think of a function is as a machine. Suppose $f$ is a function from $X$ to $Y$ and an input $x$ of $X$ is given. Imagine $f$ to be a machine that processes $x$ in a certain way to produce the output $f(x)$. This is illustrated in Figure 1.3.1.

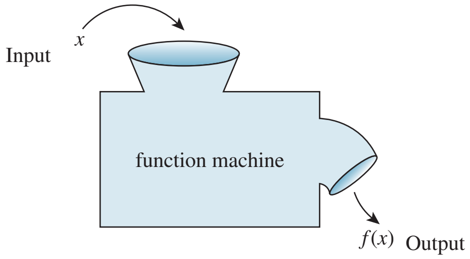


FIGURE 1.3.1

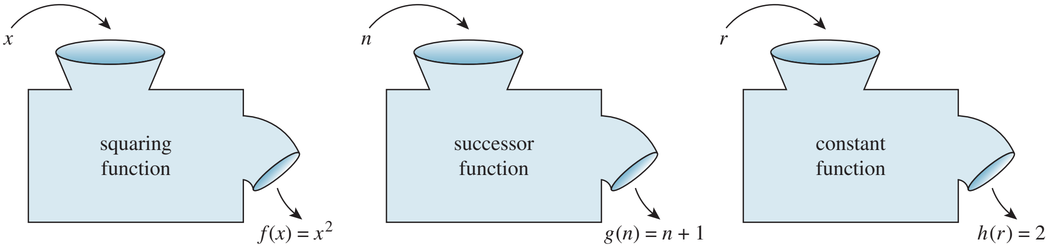
### Example 1.3.6 Functions Defined by Formulas

The **squaring function** $f$ from **R** to **R** is defined by the formula $f(x) = x^2$ for every real number $x$. This means that no matter what real number input is substituted for $x$, the output of $f$ will be the square of that number. This idea can be represented by writing $f(\square) = \square^2$. In other words, $f$ sends each real number $x$ to $x^2$, or, symbolically, $f\colon x \to x^2$. Note that the variable $x$ is a dummy variable; any other symbol could replace it, as long as the replacement is made everywhere the $x$ appears.

The **successor function** $g$ from **Z** to **Z** is defined by the formula $g(n) = n + 1$. Thus, no matter what integer is substituted for $n$, the output of $g$ will be that number plus 1: $g(\square) = \square + 1$. In other words, $g$ sends each integer $n$ to $n + 1$, or, symbolically, $g\colon n \to n + 1$.

An example of a **constant function** is the function $h$ from **Q** to **Z** defined by the formula $h(r) = 2$ for all rational numbers $r$. This function sends each rational number $r$ to 2. In other words, no matter what the input, the output is always 2: $h(\square) = 2$ or $h\colon r \to 2$.

The functions $f$, $g$, and $h$ are represented by the function machines in Figure 1.3.2.



(a) (b) (c)

FIGURE 1.3.2 ■

A function is an entity in its own right. It can be thought of as a certain relationship between sets or as an input/output machine that operates according to a certain rule. This is the reason why a function is generally denoted by a single symbol or string of symbols, such as $f$, $G$, of log, or sin.

A relation is a subset of a Cartesian product and a function is a special kind of relation. Specifically, if $f$ and $g$ are functions from a set $A$ to a set $B$, then

$$f = \{(x, y) \in A \times B \mid y = f(x)\} \quad \text{and} \quad g = \{(x, y) \in A \times B \mid y = g(x)\}.$$

It follows that

> ***f* equals *g*,** written ***f* = *g***, if, and only if, ***f*(*x*) = *g*(*x*)** for all $x$ in $A$.

### Example 1.3.7 Equality of Functions

Define functions $f$ and $g$ from **R** to **R** by the following formulas:

$$f(x) = |x| \quad \text{for every } x \in \mathbf{R}.$$
$$g(x) = \sqrt{x^2} \quad \text{for every } x \in \mathbf{R}.$$

Does $f = g$?

#### Solution

Yes. Because the absolute value of any real number equals the square root of its square, $|x| = \sqrt{x^2}$ for all $x \in \mathbf{R}$. Hence $f = g$. ■

## TEST YOURSELF

**1.** Given sets $A$ and $B$, a relation from $A$ to $B$ is ______.

**2.** A function $F$ from $A$ to $B$ is a relation from $A$ to $B$ that satisfies the following two properties:

**a.** for every element $x$ of $A$, there is ______.

**b.** for all elements $x$ in $A$ and $y$ and $z$ in $B$, if ______ then ______.

**3.** If $F$ is a function from $A$ to $B$ and $x$ is an element of $A$, then $F(x)$ is ______.

## EXERCISE SET 1.3

**1.** Let $A = \{2, 3, 4\}$ and $B = \{6, 8, 10\}$ and define a relation $R$ from $A$ to $B$ as follows: For every $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad \frac{y}{x} \text{ is an integer.}$$

**a.** Is $4\,R\,6$? Is $4\,R\,8$? Is $(3, 8) \in R$? Is $(2, 10) \in R$?
**b.** Write $R$ as a set of ordered pairs.
**c.** Write the domain and co-domain of $R$.
**d.** Draw an arrow diagram for $R$.

**2.** Let $C = D = \{-3, -2, -1, 1, 2, 3\}$ and define a relation $S$ from $C$ to $D$ as follows: For every $(x, y) \in C \times D$,

$$(x, y) \in S \quad \text{means that} \quad \tfrac{1}{x} - \tfrac{1}{y} \text{ is an integer.}$$

**a.** Is $2\,S\,2$? Is $-1\,S\,-1$? Is $(3, 3) \in S$? Is $(3, -3) \in S$?
**b.** Write $S$ as a set of ordered pairs.
**c.** Write the domain and co-domain of $S$.
**d.** Draw an arrow diagram for $S$.

**3.** Let $E = \{1, 2, 3\}$ and $F = \{-2, -1, 0\}$ and define a relation $T$ from $E$ to $F$ as follows: For every $(x, y) \in E \times F$,

$$(x, y) \in T \quad \text{means that} \quad \frac{x-y}{3} \text{ is an integer.}$$

**a.** Is $3\,T\,0$? Is $1\,T\,(-1)$? Is $(2, -1) \in T$? Is $(3, -2) \in T$?
**b.** Write $T$ as a set of ordered pairs.
**c.** Write the domain and co-domain of $T$.
**d.** Draw an arrow diagram for $T$.

**4.** Let $G = \{-2, 0, 2\}$ and $H = \{4, 6, 8\}$ and define a relation $V$ from $G$ to $H$ as follows: For every $(x, y) \in G \times H$,

$$(x, y) \in V \quad \text{means that} \quad \frac{x-y}{4} \text{ is an integer.}$$

**a.** Is $2\,V\,6$? Is $(-2)\,V\,(8)$? Is $(0, 6) \in V$? Is $(2, 4) \in V$?
**b.** Write $V$ as a set of ordered pairs.
**c.** Write the domain and co-domain of $V$.
**d.** Draw an arrow diagram for $V$.

**5.** Define a relation $S$ from $\mathbf{R}$ to $\mathbf{R}$ as follows: For every $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in S \quad \text{means that} \quad x \geq y.$$

**a.** Is $(2, 1) \in S$? Is $(2, 2) \in S$? Is $2\,S\,3$? Is $(-1)\,S\,(-2)$?
**b.** Draw the graph of $S$ in the Cartesian plane.

**6.** Define a relation $R$ from $\mathbf{R}$ to $\mathbf{R}$ as follows: For every $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$(x, y) \in R \quad \text{means that} \quad y = x^2.$$

**a.** Is $(2, 4) \in R$? Is $(4, 2) \in R$? Is $(-3)\,R\,9$? Is $9\,R\,(-3)$?
**b.** Draw the graph of $R$ in the Cartesian plane.

**7.** Let $A = \{4, 5, 6\}$ and $B = \{5, 6, 7\}$ and define relations $R$, $S$, and $T$ from $A$ to $B$ as follows: For every $(x, y) \in A \times B$:

$$(x, y) \in R \quad \text{means that} \quad x \geq y.$$
$$(x, y) \in S \quad \text{means that} \quad \frac{x-y}{2} \text{ is an integer.}$$
$$T = \{(4, 7), (6, 5), (6, 7)\}.$$

**a.** Draw arrow diagrams for $R$, $S$, and $T$.
**b.** Indicate whether any of the relations $R$, $S$, and $T$ are functions.

**8.** Let $A = \{2, 4\}$ and $B = \{1, 3, 5\}$ and define relations $U$, $V$, and $W$ from $A$ to $B$ as follows: For every $(x, y) \in A \times B$:

$$(x, y) \in U \quad \text{means that} \quad y - x > 2.$$
$$(x, y) \in V \quad \text{means that} \quad y - 1 = \tfrac{x}{2}.$$
$$W = \{(2, 5), (4, 1), (2, 3)\}.$$

**a.** Draw arrow diagrams for $U$, $V$, and $W$.
**b.** Indicate whether any of the relations $U$, $V$, and $W$ are functions.

**9.** **a.** Find all functions from $\{0, 1\}$ to $\{1\}$.
**b.** Find two relations from $\{0, 1\}$ to $\{1\}$ that are not functions.

**10.** Find four relations from $\{a, b\}$ to $\{x, y\}$ that are not functions from $\{a, b\}$ to $\{x, y\}$.

**11.** Let $A = \{0, 1, 2\}$ and let $S$ be the set of all strings over $A$. Define a relation $L$ from $S$ to $\mathbf{Z}^{nonneg}$ as follows: For every string $s$ in $S$ and every nonnegative integer $n$,

$$(s, n) \in L \quad \text{means that} \quad \text{the length of } s \text{ is } n.$$

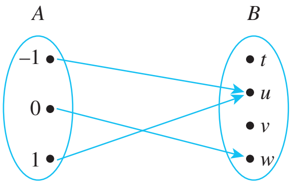
Then $L$ is a function because every string in $S$ has one and only one length. Find $L(0201)$ and $L(12)$.

**12.** Let $A = \{x, y\}$ and let $S$ be the set of all strings over $A$. Define a relation $C$ from $S$ to $S$ as follows: For all strings $s$ and $t$ in $S$,

$$(s, t) \in C \quad \text{means that} \quad t = ys.$$

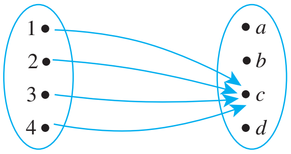
Then $C$ is a function because every string in $S$ consists entirely of $x$'s and $y$'s and adding an additional $y$ on the left creates a single new string that consists of $x$'s and $y$'s and is, therefore, also in $S$. Find $C(x)$ and $C(yyxyx)$.

**13.** Let $A = \{-1, 0, 1\}$ and $B = \{t, u, v, w\}$. Define a function $F: A \to B$ by the following arrow diagram:



**a.** Write the domain and co-domain of $F$.
**b.** Find $F(-1)$, $F(0)$, and $F(1)$.

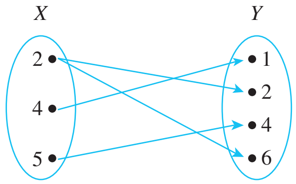
**14.** Let $C = \{1, 2, 3, 4\}$ and $D = \{a, b, c, d\}$. Define a function $G: C \to D$ by the following arrow diagram:



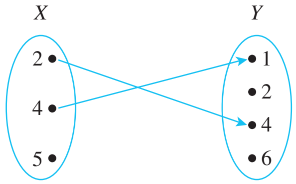
**a.** Write the domain and co-domain of $G$.
**b.** Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

**15.** Let $X = \{2, 4, 5\}$ and $Y = \{1, 2, 4, 6\}$. Which of the following arrow diagrams determine functions from $X$ to $Y$?

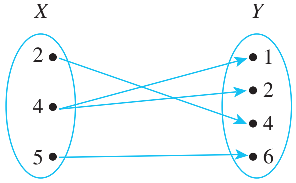
**a.**



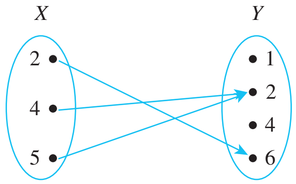
**b.**



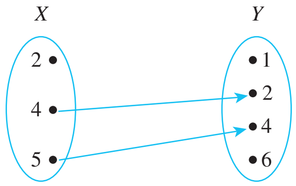
**c.**



**d.**



**e.**



**16.** Let $f$ be the squaring function defined in Example 1.3.6. Find $f(-1)$, $f(0)$, and $f\left(\frac{1}{2}\right)$.

**17.** Let $g$ be the successor function defined in Example 1.3.6. Find $g(-1000)$, $g(0)$, and $g(999)$.

**18.** Let $h$ be the constant function defined in Example 1.3.6. Find $h\left(-\frac{12}{5}\right)$, $h\left(\frac{0}{1}\right)$, and $h\left(\frac{9}{17}\right)$.

**19.** Define functions $f$ and $g$ from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For every $x \in \mathbf{R}$,

$$f(x) = 2x \quad \text{and} \quad g(x) = \frac{2x^3 + 2x}{x^2 + 1}.$$

Does $f = g$? Explain.

**20.** Define functions $H$ and $K$ from $\mathbf{R}$ to $\mathbf{R}$ by the following formulas: For every $x \in \mathbf{R}$,

$$H(x) = (x - 2)^2 \quad \text{and} \quad K(x) = (x - 1)(x - 3) + 1.$$

Does $H = K$? Explain.

## ANSWERS FOR TEST YOURSELF

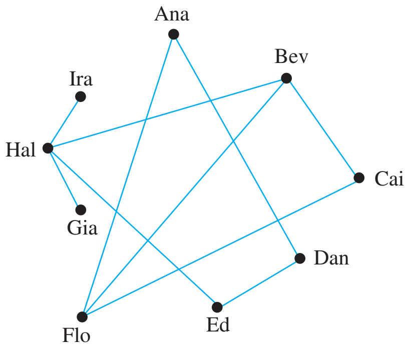
**1.** a subset of the Cartesian product $A \times B$ **2.** a. an element $y$ of $B$ such that $(x, y) \in F$ (i.e., such that $x$ is related to $y$ by $F$) b. $(x, y) \in F$ and $(x, z) \in F$; $y = z$ **3.** the unique element of $B$ that is related to $x$ by $F$

# 1.4 The Language of Graphs

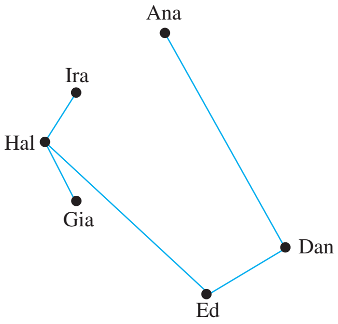
> *The whole of mathematics consists in the organization of a series of aids to the imagination in the process of reasoning.* —Alfred North Whitehead, 1861–1947

Imagine an organization that wants to set up teams of three to work on some projects. In order to maximize the number of people on each team who had previous experience working together successfully, the director asked the members to provide names of their previous partners. This information is displayed below both in a table and in a diagram.

| **Name** | **Previous Partners** |
|---|---|
| Ana | Dan, Flo |
| Bev | Cai, Flo, Hal |
| Cai | Bev, Flo |
| Dan | Ana, Ed |
| Ed | Dan, Hal |
| Flo | Cai, Bev, Ana |
| Gia | Hal |
| Hal | Gia, Ed, Bev, Ira |
| Ira | Hal |



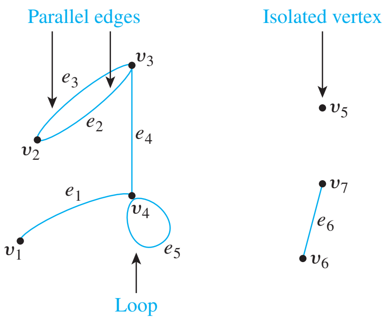
From the diagram, it is easy to see that Bev, Cai, and Flo are a group of three previous partners, and so it would be reasonable for them to form one of these teams. The drawing below shows the result when these three names are removed from the diagram.



This drawing shows that placing Hal on the same team as Ed would leave Gia and Ira on a team where they would not have a previous partner. However, if Hal is placed on a team with Gia and Ira, then the remaining team would consist of Ana, Dan, and Ed, and everyone on both teams would be working with a previous partner.

Drawings such as these are illustrations of a structure known as a *graph*. The dots are called *vertices* (plural of *vertex*) and the line segments joining vertices are called *edges*. As you can see from the first drawing, it is possible for two edges to cross at a point that is not a vertex. Note also that the type of graph described here is quite different from the "graph of an equation" or the "graph of a function."

In general, a graph consists of a set of vertices and a set of edges connecting various pairs of vertices. The edges may be straight or curved and should either connect one vertex to another or a vertex to itself, as shown below.



In this drawing, the vertices are labeled with *v*'s and the edges with *e*'s. When an edge connects a vertex to itself (as $e_5$ does), it is called a *loop*. When two edges connect the same pair of vertices (as $e_2$ and $e_3$ do), they are said to be *parallel*. It is quite possible for a vertex to be unconnected by an edge to any other vertex in the graph (as $v_5$ is), and in that case the vertex is said to be *isolated*. The formal definition of a graph follows.

**Definition**

A **graph** *G* consists of two finite sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**. The correspondence from edges to endpoints is called the **edge-endpoint function**.

An edge with just one endpoint is called a **loop**, and two or more distinct edges with the same set of endpoints are said to be **parallel**. An edge is said to **connect** its endpoints; two vertices that are connected by an edge are called **adjacent**; and a vertex that is an endpoint of a loop is said to be **adjacent to itself**.

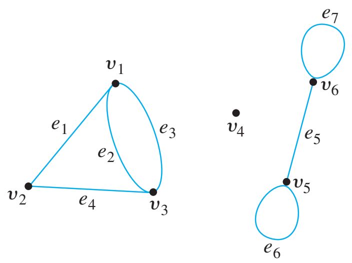
An edge is said to be **incident on** each of its endpoints, and two edges incident on the same endpoint are called **adjacent**. A vertex on which no edges are incident is called **isolated**.

Graphs have pictorial representations in which the vertices are represented by dots and the edges by line segments. A given pictorial representation uniquely determines a graph.

**Example 1.4.1**

## Terminology

Consider the following graph:



a. Write the vertex set and the edge set, and give a table showing the edge-endpoint function.

b. Find all edges that are incident on $v_1$, all vertices that are adjacent to $v_1$, all edges that are adjacent to $e_1$, all loops, all parallel edges, all vertices that are adjacent to themselves, and all isolated vertices.

### Solution

a. vertex set $= \{v_1, v_2, v_3, v_4, v_5, v_6\}$
edge set $= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
edge-endpoint function:

| Edge | Endpoints |
|---|---|
| $e_1$ | $\{v_1, v_2\}$ |
| $e_2$ | $\{v_1, v_3\}$ |
| $e_3$ | $\{v_1, v_3\}$ |
| $e_4$ | $\{v_2, v_3\}$ |
| $e_5$ | $\{v_5, v_6\}$ |
| $e_6$ | $\{v_5\}$ |
| $e_7$ | $\{v_6\}$ |

b. $e_1$, $e_2$, and $e_3$ are incident on $v_1$.
$v_2$ and $v_3$ are adjacent to $v_1$.
$e_2$, $e_3$, and $e_4$ are adjacent to $e_1$.
$e_6$ and $e_7$ are loops.
$e_2$ and $e_3$ are parallel.
$v_5$ and $v_6$ are adjacent to themselves.
$v_4$ is an isolated vertex. ■

**Note** The isolated vertex $v_4$ does not appear in the table. Although each edge of a graph must have either one or two endpoints, a vertex need not be an endpoint of an edge.

Although a given pictorial representation uniquely determines a graph, a given graph may have more than one pictorial representation. Such things as the lengths or curvatures of the edges and the relative position of the vertices on the page may vary from one pictorial representation to another.

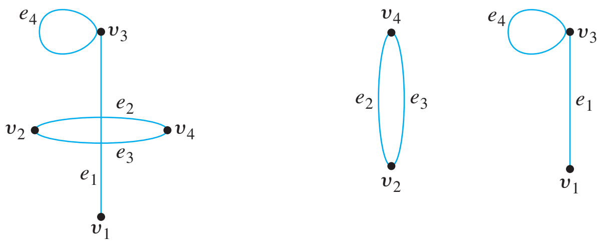
## Example 1.4.2 Drawing More Than One Picture for a Graph

Consider the graph specified as follows:

vertex set $= \{v_1, v_2, v_3, v_4\}$
edge set $= \{e_1, e_2, e_3, e_4\}$
edge-endpoint function:

| Edge | Endpoints |
|---|---|
| $e_1$ | $\{v_1, v_3\}$ |
| $e_2$ | $\{v_2, v_4\}$ |
| $e_3$ | $\{v_2, v_4\}$ |
| $e_4$ | $\{v_3\}$ |

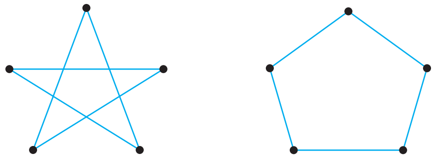
Both drawings (a) and (b) shown below are pictorial representations of this graph.



(a) (b) ■

## Example 1.4.3 Labeling Drawings to Show They Represent the Same Graph

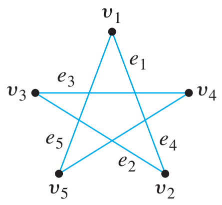
Consider the two drawings shown in Figure 1.4.1. Label vertices and edges in such a way that both drawings represent the same graph.



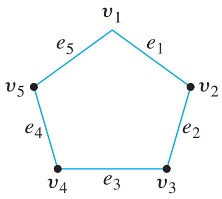
(a) (b)

FIGURE 1.4.1

**Solution** Imagine putting one end of a piece of string at the top vertex of Figure 1.4.1(a) (call this vertex $v_1$), then laying the string to the next adjacent vertex on the lower right (call this vertex $v_2$), then laying it to the next adjacent vertex on the upper left ($v_3$), and so forth, returning finally to the top vertex $v_1$. Call the first edge $e_1$, the second $e_2$, and so forth, as shown below.



Now imagine picking up the piece of string, together with its labels, and repositioning it as follows:



This is the same as Figure 1.4.1(b), so both drawings represent the graph with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$, edge set $\{e_1, e_2, e_3, e_4, e_5\}$, and edge-endpoint function as follows:

| Edge | Endpoints |
|---|---|
| $e_1$ | $\{v_1, v_2\}$ |
| $e_2$ | $\{v_2, v_3\}$ |
| $e_3$ | $\{v_3, v_4\}$ |
| $e_4$ | $\{v_4, v_5\}$ |
| $e_5$ | $\{v_5, v_1\}$ |

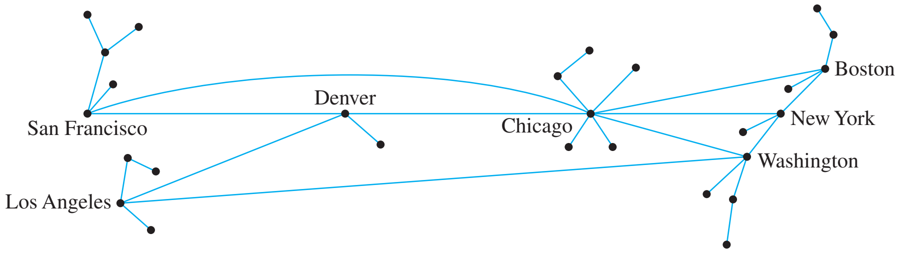
■

## *Examples of Graphs*

Graphs are a powerful problem-solving tool because they enable us to represent a complex situation with a single image that can be analyzed both visually and with the aid of a computer. A few examples follow, and others are included in the exercises.

### Example 1.4.4 Using a Graph to Represent a Network

Telephone, electric power, gas pipeline, and air transport systems can all be represented by graphs, as can computer networks—from small local area networks to the global Internet system that connects millions of computers worldwide. Questions that arise in the design of such systems involve choosing connecting edges to minimize cost, optimize a certain type of service, and so forth. A typical network, called a *hub-and-spoke model*, is shown below.

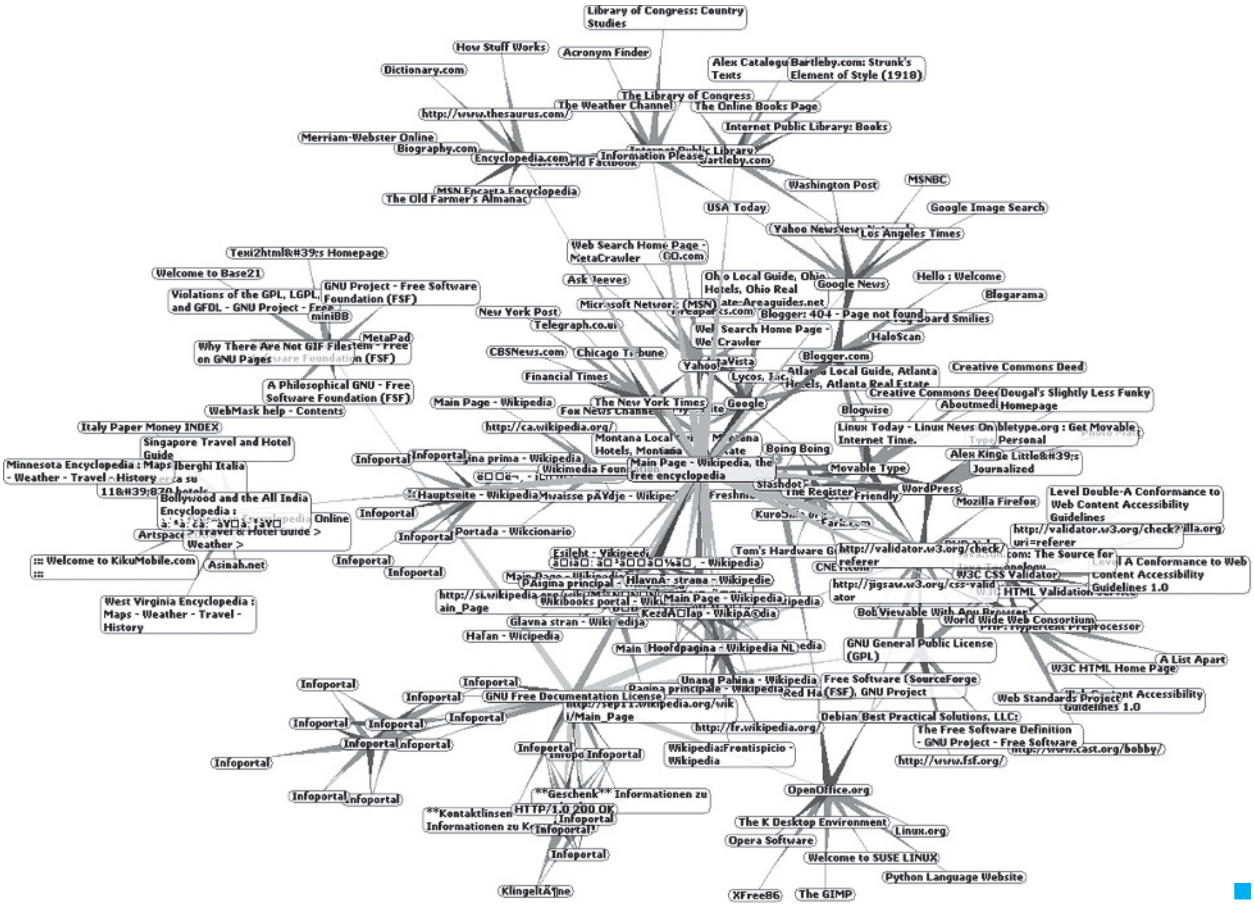


■

### Example 1.4.5 Using a Graph to Represent the World Wide Web

The World Wide Web, or Web, is a system of interlinked documents, or webpages, contained on the Internet. Users employing Web browsers, such as Internet Explorer, Chrome, Safari, and Firefox, can move quickly from one webpage to another by clicking on hyperlinks, which use versions of software called hypertext transfer protocols (HTTPs). Individuals and individual companies create the pages, which they transmit to servers that contain software capable of delivering them to those who request them through a Web browser. Because the amount of information currently on the Web is so vast, search engines, such as Google, Yahoo, and Bing, have algorithms for finding information very efficiently.

The following picture shows a minute fraction of the hyperlink connections on the Internet that radiate in and out from the Wikipedia main page.



A *directed graph* is like an (undirected) graph except that each edge is associated with an *ordered pair* of vertices rather than a *set* of vertices. Thus each edge of a directed graph can be drawn as an arrow going from the first vertex to the second vertex of the ordered pair.

**Note** Each directed graph has an associated ordinary (undirected) graph, which is obtained by ignoring the directions of the edges.

**Definition**

A **directed graph**, or **digraph**, consists of two finite sets: a nonempty set $V(G)$ of vertices and a set $D(G)$ of directed edges, where each is associated with an ordered pair of vertices called its **endpoints**. If edge $e$ is associated with the pair $(v, w)$ of vertices, then $e$ is said to be the (**directed**) **edge** from $v$ to $w$.

**Example 1.4.6**

### Using a Graph to Represent Knowledge

In many applications of artificial intelligence, a knowledge base of information is collected and represented inside a computer. Because of the way the knowledge is represented and because of the properties that govern the artificial intelligence program, the computer is not limited to retrieving data in the same form as it was entered; it can also derive new facts from the knowledge base by using certain built-in rules of inference. For example, from the knowledge that the *Los Angeles Times* is a big-city daily and that a big-city daily contains national news, an artificial intelligence program could infer that the *Los Angeles Times* contains national news. The directed graph shown in Figure 1.4.2 is a pictorial representation for a simplified knowledge base about periodical publications.

According to this knowledge base, what paper finish does the *New York Times* use?

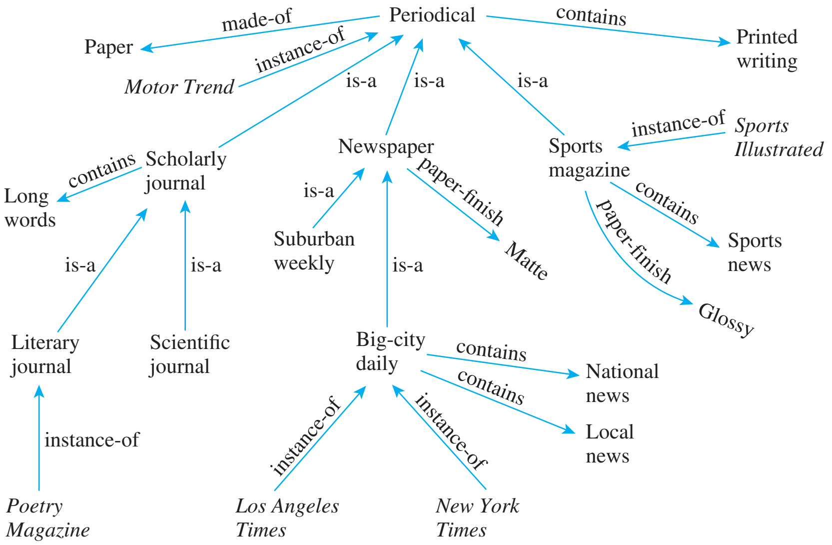


**FIGURE 1.4.2**

**Solution** The arrow going from *New York Times* to big-city daily (labeled "instance-of") shows that the *New York Times* is a big-city daily. The arrow going from big-city daily to newspaper (labeled "is-a") shows that a big-city daily is a newspaper. The arrow going from newspaper to matte (labeled "paper-finish") indicates that the paper finish on a newspaper is matte. Hence it can be inferred that the paper finish on the *New York Times* is matte. ■

## Example 1.4.7 Using a Graph to Solve a Problem: Vegetarians and Cannibals

The following is a variation of a famous puzzle often used as an example in the study of artificial intelligence. It concerns an island on which all the people are of one of two types, either vegetarians or cannibals. Initially, two vegetarians and two cannibals are on the left bank of a river. With them is a boat that can hold a maximum of two people. The aim of the puzzle is to find a way to transport all the vegetarians and cannibals to the right bank of the river. What makes this difficult is that at no time can the number of cannibals on either bank outnumber the number of vegetarians. Otherwise, disaster befalls the vegetarians!

**Solution** A systematic way to approach this problem is to introduce a notation that can indicate all possible arrangements of vegetarians, cannibals, and the boat on the banks of the river. For example, you could write ($vvc$ / $Bc$) to indicate that there are two vegetarians and one cannibal on the left bank and one cannibal and the boat on the right bank. Then ($vvccB$ /) would indicate the initial position in which both vegetarians, both cannibals, and the boat are on the left bank of the river. The aim of the puzzle is to figure out a sequence of moves to reach the position (/ $Bvvcc$) in which both vegetarians, both cannibals, and the boat are on the right bank of the river.

Construct a graph whose vertices are the various arrangements that can be reached in a sequence of legal moves starting from the initial position. Connect vertex $x$ to vertex $y$ if it is possible to reach vertex $y$ in one legal move from vertex $x$. For instance, from the initial position there are four legal moves: one vegetarian and one cannibal can take the boat to the right bank; two cannibals can take the boat to the right bank; one cannibal can take the boat to the right bank; or two vegetarians can take the boat to the right bank. You can show these by drawing edges connecting vertex (*vvccB* /) to vertices (*vc* / *Bvc*), (*vv* / *Bcc*), (*vvc* / *Bc*), and (*cc* / *Bvv*). (It might seem natural to draw directed edges rather than undirected edges from one vertex to another. The rationale for drawing undirected edges is that each legal move is reversible.) From the position (*vc* / *Bvc*), the only legal moves are to go back to (*vvccB* /) or to go to (*vvcB* / *c*). You can also show these by drawing in edges. Continue this process until finally you reach (/ *Bvvcc*). From Figure 1.4.3 it is apparent that one successful sequence of moves is (*vvccB* /) ⟶ (*vc* / *Bvc*) ⟶ (*vvcB* / *c*) ⟶ (*c* / *Bvvc*) ⟶ (*ccB* / *vv*) ⟶(/ *Bvvcc*).

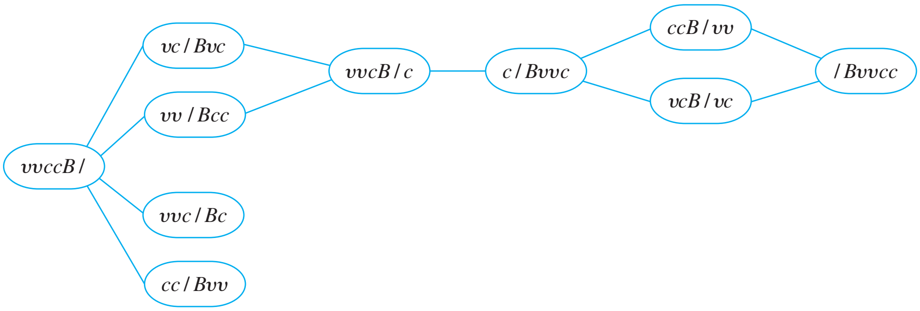
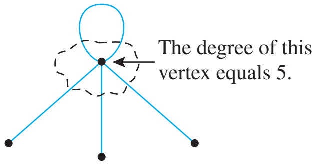


**FIGURE 1.4.3** ■

**Definition**

Let $G$ be a graph and $v$ a vertex of $G$. The **degree of $v$**, denoted **deg($v$)**, equals the number of edges that are incident on $v$, with an edge that is a loop counted twice.

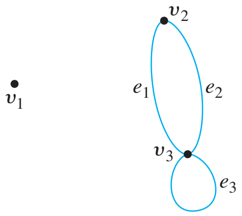
Since an edge that is a loop is counted twice, the degree of a vertex can be obtained from the drawing of a graph by counting how many end segments of edges are incident on the vertex. This is illustrated below.



**Example 1.4.8**

### Degree of a Vertex

Find the degree of each vertex of the graph $G$ shown below.



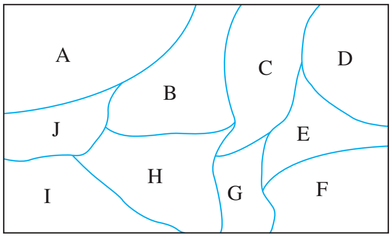
**Solution** $\deg(v_1) = 0$ since no edge is incident on $v_1$ ($v_1$ is isolated).

$\deg(v_2) = 2$ since both $e_1$ and $e_2$ are incident on $v_2$.

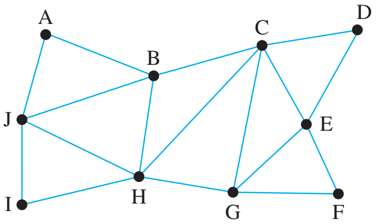
$\deg(v_3) = 4$ since $e_1$ and $e_2$ are incident on $v_3$ and the loop $e_3$ is also incident on $v_3$ (and contributes 2 to the degree of $v_3$). ■

### Example 1.4.9 Using a Graph to Color a Map

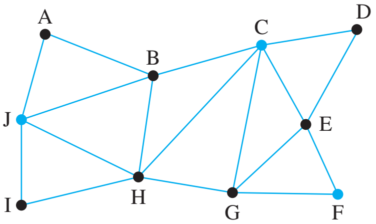
Imagine that the diagram shown below is a map with countries labeled *A–J*. Show that you can color the map so that no two adjacent countries have the same color.



**Solution** Notice that coloring the map does not depend on the sizes or shapes of the countries, but only on which countries are adjacent to which. So, to figure out a coloring, you can draw a graph, as shown below, where vertices represent countries and where edges are drawn between pairs of vertices that represent adjacent countries. Coloring the vertices of the graph will translate to coloring the countries on the map.



As you assign colors to vertices, a relatively efficient strategy is, at each stage, to focus on an uncolored vertex that has maximum degree, in other words that is connected to a maximum number of other uncolored vertices. If there is more than one such vertex, it does not matter which you choose because there are often several acceptable colorings for a given graph. For this graph, both *C* and *H* have maximum degree so you can choose one, say, *C*, and color it, say, blue. Now since *A*, *F*, *I*, and *J* are not connected to *C*, some of them may also be colored blue, and, because *J* is connected to a maximum number of others, you could start by coloring it blue. Then *F* is the only remaining vertex not connected to either *C* or *J*, so you can also color *F* blue. The drawing below shows the graph with vertices *C*, *J*, and *F* colored blue.



Since the vertices adjacent to *C*, *J*, and *F* cannot be colored blue, you can simplify the job of choosing additional colors by removing *C*, *J*, and *F* and the edges connecting them to adjacent vertices. The result is shown in Figure 1.4.4a.

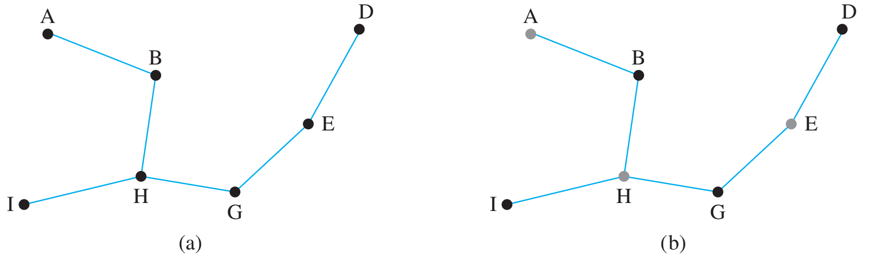
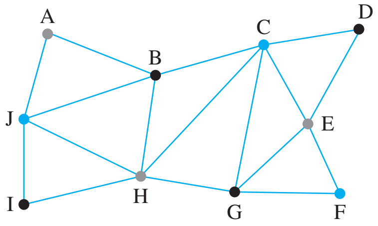
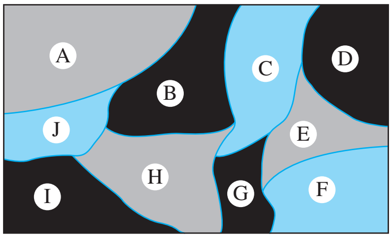


**FIGURE 1.4.4**

In the simplified graph again choose a vertex that has a maximum degree, namely *H*, and give it a second color, say, gray. Since *A*, *D*, and *E* are not connected to *H*, some of them may also be colored gray, and, because *E* is connected to a maximum number of these vertices, you could start by coloring *E* gray. Then *A* is not connected to *E*, and so you can also color *A* gray. This is shown in Figure 1.4.4b. The drawing below shows the original graph with vertices *C*, *J*, and *F* colored blue, vertices *H*, *A*, and *E*, colored gray, and the remaining vertices colored black. You can check that no two adjacent vertices have the same color.

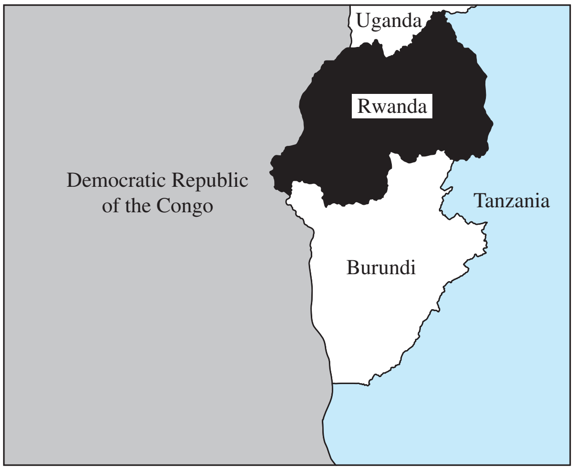


Translating the graph coloring back to the original map gives the following picture in which no two adjacent countries have the same color.



■

The final map in Example 1.4.9 was drawn with three colors. Two colors are not enough because, for example, since *B*, *C*, and *H* are all adjacent to each other, different colors must be used for all three. The following drawing shows a map of part of Central Africa that requires four colors. Take a moment to try to assign colors to the different countries so that you see why three colors are not enough.



In the mid-1800s it was conjectured that any map, however complex, could be colored with just four colors with no two adjacent regions having the same color. The conjecture is now known as the *four-color theorem* because it was finally proved true in 1976 by Kenneth Appel and Wolfgang Haken, at the University of Illinois at Urbana-Champaign. They represented maps as graphs and used an innovative and controversial technique that combined mathematical deduction with computer examination of almost 2000 special cases.

In 1950 Edward Nelson, a university student, posed the following question: How many colors are needed to create a coloring for all the points in an ordinary (Euclidean) plane so that no two points separated by a unit distance have the same color? Nelson himself found that three colors are not enough, and a fellow student, John Isbell, developed an example showing that seven colors could be used. Thus the minimum number had to be 4, 5, or 6. Over the years a number of mathematicians tried to narrow the possibilities further, but it was not until 2018 that an English biologist and amateur mathematician, Aubrey de Grey, using a combination of ingenuity and computer calculations, created an example showing that four colors are not enough. As of the publication of this book, the complete answer to Nelson's question is still unknown, but de Grey has now proved that it must be either 5 or 6.

It turns out that a variety of problems can be modeled by representing their features with a graph and solved by finding a coloring for the vertices of the graph. For example, scheduling committee meetings when members serve on more than one committee but the meetings must be held during a fixed number of time slots or scheduling final exams for a group of courses so that no student has more than two exams on any one day. See exercises 16 and 17 at the end of this section for details about these.

## TEST YOURSELF

**1.** A graph consists of two finite sets: _______ and _______, where each edge is associated with a set consisting of _______.

**2.** A loop in a graph is _______.

**3.** Two distinct edges in a graph are parallel if, and only if, _______.

**4.** Two vertices are called adjacent if, and only if, _______.

**5.** An edge is incident on _______.

**6.** Two edges incident on the same endpoint are _______.

**7.** A vertex on which no edges are incident is _______.

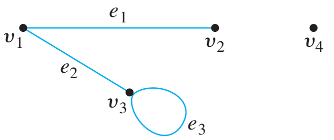
**8.** In a directed graph, each edge is associated with _______.

**9.** The degree of a vertex in a graph is ____.

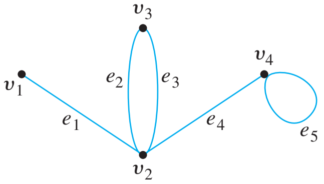
## EXERCISE SET 1.4

**In 1 and 2, graphs are represented by drawings. Define each graph formally by specifying its vertex set, its edge set, and a table giving the edge-endpoint function.**

**1.**



**2.**



**In 3 and 4, draw pictures of the specified graphs.**

**3.** Graph $G$ has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$, with edge-endpoint function as follows:

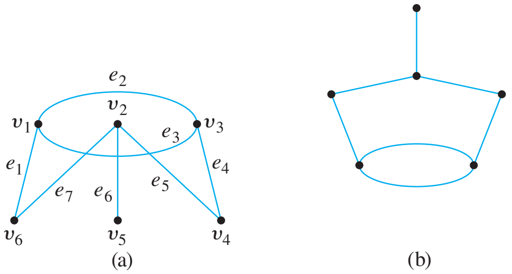
| Edge | Endpoints |
|---|---|
| $e_1$ | $\{v_1, v_2\}$ |
| $e_2$ | $\{v_1, v_2\}$ |
| $e_3$ | $\{v_2, v_3\}$ |
| $e_4$ | $\{v_2\}$ |

**4.** Graph $H$ has vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$ with edge-endpoint function as follows:

| Edge | Endpoints |
|---|---|
| $e_1$ | $\{v_1\}$ |
| $e_2$ | $\{v_2, v_3\}$ |
| $e_3$ | $\{v_2, v_3\}$ |
| $e_4$ | $\{v_1, v_5\}$ |

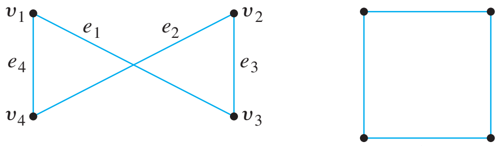
**In 5–7, show that the two drawings represent the same graph by labeling the vertices and edges of the right-hand drawing to correspond to those of the left-hand drawing.**

**5.**



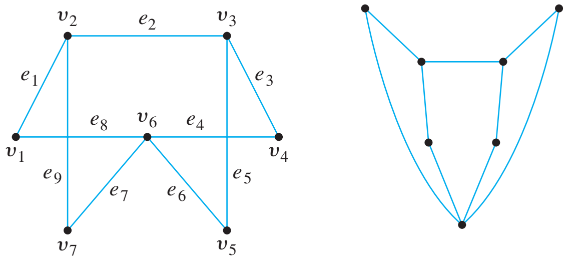
(a) (b)

**6.**



(a) (b)

**7.**

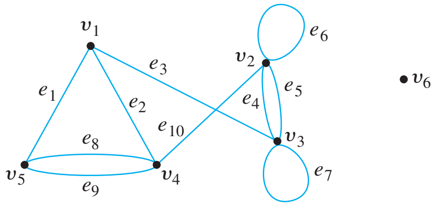


(a) (b)

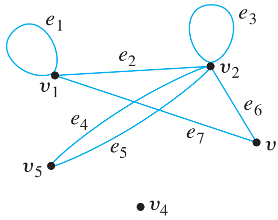
**For each of the graphs in 8 and 9:**

(i) Find all edges that are incident on $v_1$.
(ii) Find all vertices that are adjacent to $v_3$.
(iii) Find all edges that are adjacent to $e_1$.
(iv) Find all loops.
(v) Find all parallel edges.
(vi) Find all isolated vertices.
(vii) Find the degree of $v_3$.

**8.**



**9.**



**10.** Use the graph of Example 1.4.6 to determine
 **a.** whether *Sports Illustrated* contains printed writing;
 **b.** whether *Poetry Magazine* contains long words.

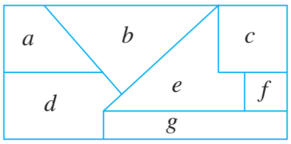
**11.** Find three other winning sequences of moves for the vegetarians and the cannibals in Example 1.4.7.

**12.** Another famous puzzle used as an example in the study of artificial intelligence seems first to have appeared in a collection of problems, *Problems for the Quickening of the Mind*, which was compiled about A.D. 775. It involves a wolf, a goat, a bag of cabbage, and a ferryman. From an initial position on the left bank of a river, the ferryman is to transport the wolf, the goat, and the cabbage to the right bank. The difficulty is that the ferryman's boat is only big enough for him to transport one object at a time, other than himself. Yet, for obvious reasons, the wolf cannot be left alone with the goat, and the goat cannot be left alone with the cabbage. How should the ferryman proceed?

**13.** Solve the vegetarians-and-cannibals puzzle for the case where there are three vegetarians and three cannibals to be transported from one side of a river to the other.

***H* 14.** Two jugs $A$ and $B$ have capacities of 3 quarts and 5 quarts, respectively. Can you use the jugs to measure out exactly 1 quart of water, while obeying the following restrictions? You may fill either jug to capacity from a water tap; you may empty the contents of either jug into a drain; and you may pour water from either jug into the other.

**15.** Imagine that the diagram shown below is a map with countries labeled $a$–$g$. Is it possible to color the map with only three colors so that no two adjacent countries have the same color? To answer this question, follow the procedure suggested by Example 1.4.9. Draw and analyze a graph in which each country is represented by a vertex and two vertices are connected by an edge if, and only if, the countries share a common border.



***H* 16.** In this exercise a graph is used to help solve a scheduling problem. Twelve faculty members in a mathematics department serve on the following committees:

Undergraduate Education: Tenner, Peterson, Kashina, Degras

Graduate Education: Hu, Ramsey, Degras, Bergen

Colloquium: Carroll, Drupieski, Au-Yeung

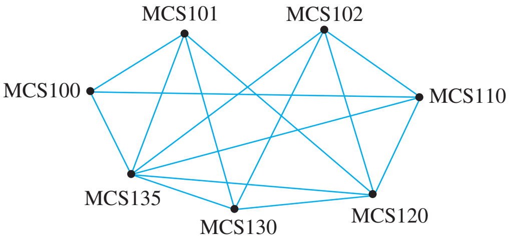
Library: Ugarcovici, Tenner, Carroll

Hiring: Hu, Drupieski, Ramsey, Peterson

Personnel: Ramsey, Wang, Ugarcovici

The committees must all meet during the first week of classes, but there are only three time slots available. Find a schedule that will allow all faculty members to attend the meetings of all committees on which they serve. To do this, represent each committee as the vertex of a graph, and draw an edge between two vertices if the two committees have a common member. Find a way to color the vertices using only three colors so that no two committees have the same color, and explain how to use the result to schedule the meetings.

**17.** A department wants to schedule final exams so that no student has more than one exam on any given day. The vertices of the graph below show the courses that are being taken by more than one student, with an edge connecting two vertices if there is a student in both courses. Find a way to color the vertices of the graph with only four colors so that no two adjacent vertices have the same color and explain how to use the result to schedule the final exams.

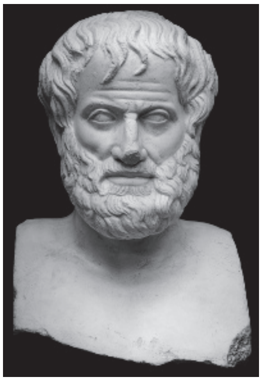


## ANSWERS FOR TEST YOURSELF

**1.** a finite, nonempty set of vertices; a finite set of edges; one or two vertices called its endpoints **2.** an edge with a single endpoint **3.** they have the same set of endpoints **4.** they are connected by an edge **5.** each of its endpoints **6.** adjacent **7.** isolated **8.** an ordered pair of vertices called its endpoints **9.** the number of edges that are incident on the vertex, with an edge that is a loop counted twice

## CHAPTER 2

# THE LOGIC OF COMPOUND STATEMENTS



Mohamed Osama/Alamy Stock Photo

*Aristotle*
*(384 B.C.–322 B.C.)*

The first great treatises on logic were written by the Greek philosopher Aristotle. They were a collection of rules for deductive reasoning that were intended to serve as a basis for the study of every branch of knowledge. In the seventeenth century, the German philosopher and mathematician Gottfried Leibniz conceived the idea of using symbols to mechanize the process of deductive reasoning in much the same way that algebraic notation had mechanized the process of reasoning about numbers and their relationships. Leibniz's idea was realized in the nineteenth century by the English mathematicians George Boole and Augustus De Morgan, who founded the modern subject of symbolic logic. With research continuing to the present day, symbolic logic has provided, among other things, the theoretical basis for many areas of computer science such as digital logic circuit design (see Sections 2.4 and 2.5), relational database theory (see Section 8.1), automata theory and computability (see Section 7.4 and Chapter 12), and artificial intelligence (see Sections 3.3, 10.1, and 10.5).

## 2.1 Logical Form and Logical Equivalence

*Logic is a science of the necessary laws of thought, without which no employment of the understanding and the reason takes place.* —Immanuel Kant, 1785

An argument is a sequence of statements aimed at demonstrating the truth of an assertion. The assertion at the end of the sequence is called the *conclusion*, and the preceding statements are called *premises*. To have confidence in the conclusion that you draw from an argument, you must be sure that the premises are acceptable on their own merits or follow from other statements that are known to be true.

In logic, the form of an argument is distinguished from its content. Logical analysis won't help you determine the intrinsic merit of an argument's content, but it will help you analyze an argument's form to determine whether the truth of the conclusion follows *necessarily* from the truth of the premises. For this reason logic is sometimes defined as the science of necessary inference or the science of reasoning.

Consider the following two arguments. They have very different content but their logical form is the same. To help make this clear, we use letters like $p$, $q$, and $r$ to represent component sentences; we let the expression "not $p$" refer to the sentence "It is not the case that $p$"; and we let the symbol $\therefore$ stand for the word "therefore."

Argument 1

$$\text{If } \overbrace{\text{the bell rings}}^{p} \text{ or } \overbrace{\text{the flag drops}}^{q}\text{, then } \overbrace{\text{the race is over}}^{r}.$$

$$\therefore \text{ If } \overbrace{\text{the race is not over}}^{\text{not } r}\text{, then } \overbrace{\text{the bell hasn't rung}}^{\text{not } p} \text{ and } \overbrace{\text{the flag hasn't dropped}}^{\text{not } q}.$$

Argument 2

$$\text{If } \overbrace{x = 2}^{p} \text{ or } \overbrace{x = -2}^{q}\text{, then } \overbrace{x^2 = 4}^{r}.$$

$$\therefore \text{ If } \overbrace{x^2 \neq 4}^{\text{not } r}\text{, then } \overbrace{x \neq 2}^{\text{not } p} \text{ and } \overbrace{x \neq -2}^{\text{not } q}.$$

The common form of the arguments is

If $p$ or $q$, then $r$.
$\therefore$ If not $r$, then not $p$ and not $q$.

In exercise 10 in Section 2.3 you will show that this form of argument is *valid* in the sense that *if* its assumptions are true, then its conclusion must also be true.

**Example 2.1.1**

### Identifying Logical Form

Fill in the blanks below so that argument (b) has the same form as argument (a). Then represent the common form of the arguments using letters to stand for component sentences.

a. If Jane is a math major or Jane is a computer science major, then Jane will take Math 150.
Jane is a computer science major.
Therefore, Jane will take Math 150.

b. If logic is easy or $\underline{\quad (1) \quad}$, then $\underline{\quad (2\,) \quad}$.
I will study hard.
Therefore, I will get an A in this course.

### Solution

1. I (will) study hard.
2. I will get an A in this course.

*Common form:* If $p$ or $q$, then $r$.
$q$.
Therefore, $r$. ■

## Statements

Most of the definitions of formal logic have been developed so that they agree with the natural or intuitive logic used by people who have been educated to think clearly and use language carefully. The differences that exist between formal and intuitive logic are necessary to avoid ambiguity and obtain consistency.

In any mathematical theory, new terms are defined by using those that have been previously defined. However, this process has to start somewhere. A few initial terms necessarily remain undefined. In logic, the words *sentence*, *true*, and *false* are the initial undefined terms.

**Definition**

A **statement** (or **proposition**) is a sentence that is true or false but not both.

For example, “Two plus two equals four” and “Two plus two equals five” are both statements, the first because it is true and the second because it is false. On the other hand, the truth or falsity of

$$x^2 + 2 = 11$$

depends on the value of $x$. For some values of $x$, it is true ($x = 3$ and $x = -3$), whereas for other values it is false. Similarly, the truth or falsity of

$$x + y > 0$$

depends on the values of $x$ and $y$. For instance, when $x = -1$ and $y = 2$ it is true, whereas when $x = -1$ and $y = 1$ it is false. In Section 3.1 we will discuss ways to transform sentences of these forms into statements.

## *Compound Statements*

**Note** $\sim p$ means "not $p$"
$p \wedge q$ means "$p$ and $q$"
$p \vee q$ means "$p$ or $q$"

We now introduce three symbols that are used to build more complicated logical expressions out of simpler ones. The symbol $\sim$ denotes *not*, $\wedge$ denotes *and*, and $\vee$ denotes *or*. Given a statement $p$, the sentence "$\sim p$" is read "not $p$" or "It is not the case that $p$." In some computer languages the symbol $\neg$ is used in place of $\sim$. Given another statement $q$, the sentence "$p \wedge q$" is read "$p$ and $q$." The sentence "$p \vee q$" is read "$p$ or $q$."

In expressions that include the symbol $\sim$ as well as $\wedge$ or $\vee$, the **order of operations** specifies that $\sim$ is performed first. For instance, $\sim p \wedge q = (\sim p) \wedge q$. In logical expressions, as in ordinary algebraic expressions, the order of operations can be overridden through the use of parentheses. Thus $\sim(p \wedge q)$ represents the negation of the conjunction of $p$ and $q$. In this, as in most treatments of logic, the symbols $\wedge$ and $\vee$ are considered coequal in order of operation, and an expression such as $p \wedge q \vee r$ is considered ambiguous. This expression must be written as either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$ to have meaning.

A variety of English words translate into logic as $\wedge$, $\vee$, or $\sim$. For instance, the word *but* translates the same as *and* when it links two independent clauses, as in "Jim is tall but he is not heavy." Generally, the word *but* is used in place of *and* when the part of the sentence that follows is, in some way, unexpected. Another example involves the words *neither-nor*. When Shakespeare wrote, "Neither a borrower nor a lender be," he meant, "Do not be a borrower and do not be a lender." So if $p$ and $q$ are statements, then

| | | |
|---|---|---|
| $p$ but $q$ | means | $p$ and $q$ |
| neither $p$ nor $q$ | means | $\sim p$ and $\sim q$. |

### Example 2.1.2 Translating from English to Symbols: *But* and *Neither-Nor*

Write each of the following sentences symbolically, letting $h$ = "It is hot" and $s$ = "It is sunny."

a. It is not hot but it is sunny.

b. It is neither hot nor sunny.

**Solution**

a. The given sentence is equivalent to "It is not hot and it is sunny," which can be written symbolically as $\sim h \wedge s$.

b. To say it is neither hot nor sunny means that it is not hot and it is not sunny. Therefore, the given sentence can be written symbolically as $\sim h \wedge \sim s$. ■

The notation for inequalities involves *and* and *or* statements. For instance, if $x$, $a$, and $b$ are particular real numbers, then

$$\boxed{\begin{array}{rcccc} x \le a & \text{means} & x < a & \text{or} & x = a \\ a \le x \le b & \text{means} & a \le x & \text{and} & x \le b. \end{array}}$$

**Note** The point of specifying $x$, $a$, and $b$ to be *particular* real numbers is to ensure that sentences such as "$x < a$" and "$x \ge b$" are either true or false and hence that they are statements.

Note that the inequality $2 \le x \le 1$ is not satisfied by any real numbers because

$$\boxed{\begin{array}{ccccc} 2 \le x \le 1 & \text{means} & 2 \le x & \text{and} & x \le 1, \end{array}}$$

and this is false no matter what number $x$ happens to be.

**Example 2.1.3**

### *And*, *Or*, and Inequalities

Suppose $x$ is a particular real number. Let $p$, $q$, and $r$ symbolize "$0 < x$," "$x < 3$," and "$x = 3$," respectively. Write the following inequalities symbolically:

a. $x \le 3$   b. $0 < x < 3$   c. $0 < x \le 3$

**Solution**

a. $q \vee r$   b. $p \wedge q$   c. $p \wedge (q \vee r)$ ■

## *Truth Values*

In Examples 2.1.2 and 2.1.3 we built compound sentences out of component statements and the terms *not*, *and*, and *or*. If such sentences are to be statements, however, they must have well-defined **truth values**—they must be either true or false. We now define such compound sentences as statements by specifying their truth values in terms of the statements that compose them.

**Note** Think of negation like this:

> The negation of a statement is a statement that exactly expresses what it would mean for the statement to be false.

**Definition**

If $p$ is a statement variable, the **negation** of $p$ is "not $p$" or "It is not the case that $p$" and is denoted $\sim p$. It has opposite truth value from $p$: if $p$ is true, $\sim p$ is false; if $p$ is false, $\sim p$ is true.

The truth values for negation are summarized in a *truth table*.

Truth Table for $\sim p$

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

In ordinary language the sentence "It is hot and it is sunny" is understood to be true when both conditions—being hot and being sunny—are satisfied. If it is hot but not sunny, or sunny but not hot, or neither hot nor sunny, the sentence is understood to be false. The formal definition of truth values for an *and* statement agrees with this general understanding.

**Definition**

If $p$ and $q$ are statement variables, the **conjunction** of $p$ and $q$ is "$p$ and $q$," denoted $p \wedge q$. It is true when, and only when, both $p$ and $q$ are true. If either $p$ or $q$ is false, or if both are false, $p \wedge q$ is false.

The truth values for conjunction can also be summarized in a truth table. The table is obtained by considering the four possible combinations of truth values for $p$ and $q$. Each combination is displayed in one row of the table; the corresponding truth value for the whole statement is placed in the right-most column of that row. Note that the only row containing a T is the first one because an *and* statement is true only when both components are true.

**Note** The only way for an *and* statement to be true is for both components to be true. So in the truth table for an *and* statement the first row is the only row with a *T*.

Truth Table for $p \wedge q$

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

The order of truth values for $p$ and $q$ in the table above is TT, TF, FT, FF. It is not absolutely necessary to write the truth values in this order, although it is customary to do so. So please use this order for all truth tables involving two statement variables. Example 2.1.5 shows the standard order for truth tables that involve three statement variables.

In the case of disjunction—statements of the form "$p$ or $q$"—intuitive logic offers two alternative interpretations. In ordinary language *or* is sometimes used in an exclusive sense ($p$ or $q$ but not both) and sometimes in an inclusive sense ($p$ or $q$ or both). A waiter who says you may have "coffee, tea, or milk" uses the word *or* in an exclusive sense: Extra payment is generally required if you want more than one beverage. On the other hand, a waiter who offers "cream or sugar" uses the word *or* in an inclusive sense: You are entitled to both cream and sugar if you wish to have them.

Mathematicians and logicians avoid possible ambiguity about the meaning of the word *or* by understanding it to mean the inclusive "and/or." The symbol $\vee$ comes from the Latin word *vel*, which means *or* in its inclusive sense. To express the exclusive *or*, the phrase *p or q but not both* is used.

**Note** The statement "$2 \leq 2$" means that 2 is less than 2 or 2 equals 2. It is true because $2 = 2$.

**Definition**

If $p$ and $q$ are statement variables, the **disjunction** of $p$ and $q$ is "$p$ or $q$," denoted $p \vee q$. It is true when either $p$ is true, or $q$ is true, or both $p$ and $q$ are true; it is false only when both $p$ and $q$ are false.

Here is the truth table for disjunction:

**Note** The only way for an *or* statement to be false is for both components to be false. So in the truth table for an *or* statement the last row is the only row with an *F*.

Truth Table for $p \vee q$

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

## Evaluating the Truth of More General Compound Statements

**Note** Java, C, and C++ use the following notations:

| | |
|---|---|
| $\sim$ | ! |
| $\wedge$ | && |
| $\vee$ | \|\| |

Now that truth values have been assigned to $\sim p$, $p \wedge q$, and $p \vee q$, consider the question of assigning truth values to more complicated expressions such as $\sim p \vee q$, $(p \vee q) \wedge \sim(p \wedge q)$, and $(p \wedge q) \vee r$. Such expressions are called *statement forms* (or *propositional forms*). The close relationship between statement forms and *Boolean expressions* is discussed in Section 2.4.

**Definition**

A **statement form** (or **propositional form**) is an expression made up of statement variables (such as $p$, $q$, and $r$) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables. The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

To compute the truth values for a statement form, follow rules similar to those used to evaluate algebraic expressions. For each combination of truth values for the statement variables, first evaluate the expressions within the innermost parentheses, then evaluate the expressions within the next innermost set of parentheses, and so forth, until you have the truth values for the complete expression.

### Example 2.1.4 Truth Table for *Exclusive Or*

**Note** Exclusive *or* is often symbolized as $p \oplus q$ or $p$ XOR $q$.

Construct the truth table for the statement form $(p \vee q) \wedge \sim(p \wedge q)$. Note that when *or* is used in its exclusive sense, the statement "$p$ or $q$" means "$p$ or $q$ but not both" or "$p$ or $q$ and not both $p$ and $q$," which translates into symbols as $(p \vee q) \wedge \sim(p \wedge q)$.

**Solution** Set up columns labeled $p$, $q$, $p \vee q$, $p \wedge q$, $\sim(p \wedge q)$, and $(p \vee q) \wedge \sim(p \wedge q)$. Fill in the $p$ and $q$ columns with all the logically possible combinations of T's and F's. Then use the truth tables for $\vee$ and $\wedge$ to fill in the $p \vee q$ and $p \wedge q$ columns with the appropriate truth values. Next fill in the $\sim(p \wedge q)$ column by taking the opposites of the truth values for $p \wedge q$. For example, the entry for $\sim(p \wedge q)$ in the first row is F because in the first row the truth value of $p \wedge q$ is T. Finally, fill in the $(p \vee q) \wedge \sim(p \wedge q)$ column by considering the truth values for an *and* statement together with the truth values for $p \vee q$ and $\sim(p \wedge q)$. Since an *and* statement is true only when both components are true and since rows 2 and 3 are the only two rows where both $p \vee q$ and $\sim(p \wedge q)$ are true, put T in rows 2 and 3 and F in the remaining rows.

**Note** To fill out a truth table for an *and* statement, first put a T in each row where both components are true; then put an F in each of the remaining rows.

Truth Table for *Exclusive Or:* $(p \vee q) \wedge \sim(p \wedge q)$

| $p$ | $q$ | $p \vee q$ | $p \wedge q$ | $\sim(p \wedge q)$ | $(p \vee q) \wedge \sim(p \wedge q)$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | F | T | F | T | T |
| F | T | T | F | T | T |
| F | F | F | F | T | F |

■

**Example 2.1.5**

### Truth Table for $(p \wedge q) \vee \sim r$

Construct a truth table for the statement form $(p \wedge q) \vee \sim r$.

**Note** To fill out a truth table for an *or* statement, first put an F in each row where both components are false; then put a T in each of the remaining rows.

**Solution** Make columns headed $p$, $q$, $r$, $p \wedge q$, $\sim r$, and $(p \wedge q) \vee \sim r$. Enter the eight logically possible combinations of truth values for $p$, $q$, and $r$ in the three left-most columns. Then fill in the truth values for $p \wedge q$ and for $\sim r$. Complete the table by considering the truth values for $(p \wedge q)$ and for $\sim r$ and the definition of an *or* statement. Since an *or* statement is false only when both components are false, the only rows in which the entry is F are the third, fifth, and seventh rows because those are the only rows in which the expressions $p \wedge q$ and $\sim r$ are both false. The entry for all the other rows is T.

| $p$ | $q$ | $r$ | $p \wedge q$ | $\sim r$ | $(p \wedge q) \vee \sim r$ |
|---|---|---|---|---|---|
| T | T | T | T | F | T |
| T | T | F | T | T | T |
| T | F | T | F | F | F |
| T | F | F | F | T | T |
| F | T | T | F | F | F |
| F | T | F | F | T | T |
| F | F | T | F | F | F |
| F | F | F | F | T | T |

■

The essential point about assigning truth values to compound statements is that it allows you—using logic alone—to judge the truth of a compound statement on the basis of your knowledge of the truth of its component parts. Logic does not help you determine the truth or falsity of the component statements. Rather, logic helps link these separate pieces of information together into a coherent whole.

## *Logical Equivalence*

The statements

6 is greater than 2 and 2 is less than 6

are two different ways of saying the same thing. Why? Because of the definition of the phrases *greater than* and *less than*. By contrast, although the statements

(1) Dogs bark and cats meow and (2) Cats meow and dogs bark

are also two different ways of saying the same thing, the reason has nothing to do with the definition of the words. It has to do with the logical form of the statements. Any two statements whose logical forms are related in the same way as (1) and (2) would either both be true or both be false. You can see this by examining the following truth table, where the statement variables $p$ and $q$ are substituted for the component statements "Dogs bark" and "Cats meow," respectively. The table shows that for each combination of truth values for $p$ and $q$, $p \wedge q$ is true when, and only when, $q \wedge p$ is true. In such a case, the statement forms are called *logically equivalent*, and we say that (1) and (2) are *logically equivalent statements.*

| $p$ | $q$ | $p \wedge q$ | $q \wedge p$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | F | F |
| F | F | F | F |

$p \wedge q$ and $q \wedge p$ always have the same truth values, so they are logically equivalent

## Definition

Two *statement forms* are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms $P$ and $Q$ is denoted by writing $P \equiv Q$.

Two *statements* are called **logically equivalent** if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

## Testing Whether Two Statement Forms *P* and *Q* Are Logically Equivalent

1. Construct a truth table with one column for the truth values of $P$ and another column for the truth values of $Q$.
2. Check each combination of truth values of the statement variables to see whether the truth value of $P$ is the same as the truth value of $Q$.
   a. If in each row the truth value of $P$ is the same as the truth value of $Q$, then $P$ and $Q$ are logically equivalent.
   b. If in some row $P$ has a different truth value from $Q$, then $P$ and $Q$ are not logically equivalent.

**Example 2.1.6**

### Double Negative Property: $\sim(\sim p) \equiv p$

Construct a truth table to show that the negation of the negation of a statement is logically equivalent to the statement, annotating the table with a sentence of explanation.

### Solution

| $p$ | $\sim p$ | $\sim(\sim p)$ |
|---|---|---|
| T | F | T |
| F | T | F |

$p$ and $\sim(\sim p)$ always have the same truth values, so they are logically equivalent

■

There are two ways to show that statement forms $P$ and $Q$ are *not* logically equivalent. As indicated previously, one is to use a truth table to find rows for which their truth values differ. The other way is to find concrete statements for each of the two forms, one of which is true and the other of which is false. The next example illustrates both of these ways.

**Example 2.1.7**

## Showing Nonequivalence

Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

### Solution

a. This method uses a truth table annotated with a sentence of explanation.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim(p \wedge q)$ | | $\sim p \wedge \sim q$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | | F |
| T | F | F | T | F | T | $\neq$ | F |
| F | T | T | F | F | T | $\neq$ | F |
| F | F | T | T | F | T | | T |

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

b. This method uses an example to show that $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent. Let $p$ be the statement "$0 < 1$" and let $q$ be the statement "$1 < 0$." Then

$$\sim(p \wedge q) \quad \text{is} \quad \text{“It is not the case that both } 0 < 1 \text{ and } 1 < 0,\text{”}$$

which is true. On the other hand,

$$\sim p \wedge \sim q \quad \text{is} \quad \text{“}0 \not< 1 \quad \text{and} \quad 1 \not< 0,\text{”}$$

which is false. This example shows that there are concrete statements you can substitute for $p$ and $q$ to make one of the statement forms true and the other false. Therefore, the statement forms are not logically equivalent. ■

**Example 2.1.8**

## Negations of *And* and *Or*: De Morgan's Laws

For the statement "John is tall and Jim is redheaded" to be true, both components must be true. So for the statement to be false, one or both components must be false. Thus the negation can be written as "John is not tall or Jim is not redheaded." In general, the negation of the conjunction of two statements is logically equivalent to the disjunction of their negations. That is, statements of the forms $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent. Check this using truth tables.

### Solution

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim(p \wedge q)$ | $\sim p \vee \sim q$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T |

$\sim(p \wedge q)$ and $\sim p \vee \sim q$ always have the same truth values, so they are logically equivalent

Symbolically,

$$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q.}$$

Paul Fearn/Alamy Stock Photo

*Augustus De Morgan (1806–1871)*

In the exercises at the end of this section you are asked to show the analogous law that the negation of the disjunction of two statements is logically equivalent to the conjunction of their negations:

$$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q.}$$

■

The two logical equivalences of Example 2.1.8 are known as **De Morgan's laws** of logic in honor of Augustus De Morgan, who was the first to state them in formal mathematical terms.

### De Morgan's Laws

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

**Example 2.1.9**

## Applying De Morgan's Laws

Write negations for each of the following statements:

a. John is 6 feet tall and he weighs at least 200 pounds.

b. The bus was late or Tom's watch was slow.

### Solution

a. John is not 6 feet tall or he weighs less than 200 pounds.

b. The bus was not late and Tom's watch was not slow.

Since the statement "neither $p$ nor $q$" means the same as "$\sim p$ and $\sim q$," an alternative answer for (b) is "Neither was the bus late nor was Tom's watch slow." ■

If $x$ is a particular real number, saying that $x$ is not less than 2 ($x \nless 2$) means that $x$ does not lie to the left of 2 on the number line. This is equivalent to saying that either $x = 2$ or $x$ lies to the right of 2 on the number line ($x = 2$ or $x > 2$). Hence,

$$x \nless 2 \quad \text{is equivalent to} \quad x \geq 2.$$

Pictorially,

If $x \nless 2$, then $x$ lies in the shaded region.

Similarly,

$$\begin{aligned} x \ngtr 2 &\quad \text{is equivalent to} \quad x \leq 2, \\ x \nleq 2 &\quad \text{is equivalent to} \quad x > 2, \text{ and} \\ x \ngeq 2 &\quad \text{is equivalent to} \quad x < 2. \end{aligned}$$

**Example 2.1.10**

### Inequalities and De Morgan's Laws

Use De Morgan's laws to write the negation of $-1 < x \leq 4$.

**Solution** The given statement is equivalent to

$$-1 < x \quad \text{and} \quad x \leq 4.$$

By De Morgan's laws, the negation is

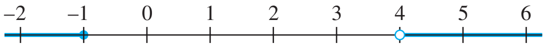
$$-1 \nless x \quad \text{or} \quad x \nleq 4,$$

**Caution!** The negation of $-1 < x \leq 4$ is *not* $-1 \nless x \nleq 4$. It is also not $-1 \geq x > 4$.

which is equivalent to

$$-1 \geq x \quad \text{or} \quad x > 4.$$

Pictorially, if $-1 \geq x$ or $x > 4$, then $x$ lies in the shaded region of the number line, as shown below.



■

De Morgan's laws are frequently used in writing computer programs. For instance, suppose you want your program to delete all files modified outside a certain range of dates, say from date 1 through date 2 inclusive. You would use the fact that

$$\sim(date1 \leq file\_modification\_date \leq date2)$$

is equivalent to

$$(file\_modification\_date < date1) \quad \text{or} \quad (date2 < file\_modification\_date).$$

**Example 2.1.11**

### A Cautionary Example

According to De Morgan's laws, the negation of

$p$: Jim is tall and Jim is thin

is $\sim p$: Jim is not tall or Jim is not thin

because the negation of an *and* statement is the *or* statement in which the two components are negated.

Unfortunately, a potentially confusing aspect of the English language can arise when you are taking negations of this kind. Note that statement $p$ can be written more compactly as

$$p': \text{Jim is tall and thin.}$$

When it is so written, another way to negate it is

$$\sim(p'): \text{Jim is not tall and thin.}$$

But in this form the negation looks like an *and* statement. Doesn't that violate De Morgan's laws?

Actually no violation occurs. The reason is that in formal logic the words *and* and *or* are allowed only between complete statements, not between sentence fragments. So when you apply De Morgan's laws, you must have complete statements on either side of each *and* and on either side of each *or*. ■

**Caution!** Although the laws of logic are extremely useful, they should be used as an *aid* to thinking, not as a mechanical substitute for it.

## *Tautologies and Contradictions*

It has been said that all of mathematics reduces to tautologies. Although this is formally true, most working mathematicians think of their subject as having substance as well as form. Nonetheless, an intuitive grasp of basic logical tautologies is part of the equipment of anyone who reasons with mathematics.

**Definition**

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement**.

A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is a **contradictory statement**.

According to this definition, the truth of a tautological statement and the falsity of a contradictory statement are due to the logical structure of the statements themselves and are independent of the meanings of the statements.

### Example 2.1.12 Tautologies and Contradictions

Show that the statement form $p \vee \sim p$ is a tautology and that the statement form $p \wedge \sim p$ is a contradiction.

**Solution**

| $p$ | $\sim p$ | $p \vee \sim p$ | $p \wedge \sim p$ |
|---|---|---|---|
| T | F | T | F |
| F | T | T | F |

↑ all T's, so $p \vee \sim p$ is a tautology

↑ all F's, so $p \wedge \sim p$ is a contradiction

■

**Example 2.1.13** **Logical Equivalence Involving Tautologies and Contradictions**

If $\mathbf{t}$ is a tautology and $\mathbf{c}$ is a contradiction, show that $p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$.

**Solution**

| $p$ | $\mathbf{t}$ | $p \wedge \mathbf{t}$ | $p$ | $\mathbf{c}$ | $p \wedge \mathbf{c}$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| F | T | F | F | F | F |

same truth values, so $p \wedge \mathbf{t} \equiv p$

same truth values, so $p \wedge \mathbf{c} \equiv \mathbf{c}$

■

## *Summary of Logical Equivalences*

Knowledge of logically equivalent statements is very useful for constructing arguments. It often happens that it is difficult to see how a conclusion follows from one form of a statement, whereas it is easy to see how it follows from a logically equivalent form of the statement. A number of logical equivalences are summarized in Theorem 2.1.1 for future reference.

**Theorem 2.1.1 Logical Equivalences**

Given any statement variables $p$, $q$, and $r$, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

| | | | |
|---|---|---|---|
| 1. | *Commutative laws:* | $p \wedge q \equiv q \wedge p$ | $p \vee q \equiv q \vee p$ |
| 2. | *Associative laws:* | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ |
| 3. | *Distributive laws:* | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ |
| 4. | *Identity laws:* | $p \wedge \mathbf{t} \equiv p$ | $p \vee \mathbf{c} \equiv p$ |
| 5. | *Negation laws:* | $p \vee \sim p \equiv \mathbf{t}$ | $p \wedge \sim p \equiv \mathbf{c}$ |
| 6. | *Double negative law:* | $\sim(\sim p) \equiv p$ | |
| 7. | *Idempotent laws:* | $p \wedge p \equiv p$ | $p \vee p \equiv p$ |
| 8. | *Universal bound laws:* | $p \vee \mathbf{t} \equiv \mathbf{t}$ | $p \wedge \mathbf{c} \equiv \mathbf{c}$ |
| 9. | *De Morgan's laws:* | $\sim(p \wedge q) \equiv \sim p \vee \sim q$ | $\sim(p \vee q) \equiv \sim p \wedge \sim q$ |
| 10. | *Absorption laws:* | $p \vee (p \wedge q) \equiv p$ | $p \wedge (p \vee q) \equiv p$ |
| 11. | *Negations of* **t** *and* **c**: | $\sim \mathbf{t} \equiv \mathbf{c}$ | $\sim \mathbf{c} \equiv \mathbf{t}$ |

The proofs of laws 4 and 6, the first parts of laws 1 and 5, and the second part of law 9 have already been given as examples in the text. Proofs of the other parts of the theorem are left as exercises. In fact, it can be shown that the first five laws of Theorem 2.1.1 form a core from which the other laws can be derived. The first five laws are the axioms for a mathematical structure known as a Boolean algebra, which is discussed in Section 6.4.

The equivalences of Theorem 2.1.1 are general laws of thought that occur in all areas of human endeavor. They can also be used in a formal way to rewrite complicated statement forms more simply.

### Example 2.1.14 Simplifying Statement Forms

Use Theorem 2.1.1 to verify the logical equivalence

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p.$$

**Solution** Use the laws of Theorem 2.1.1 to replace sections of the statement form on the left by logically equivalent expressions. Each time you do this, you obtain a logically equivalent statement form. Continue making replacements until you obtain the statement form on the right.

$$\begin{aligned}
\sim(\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) && \text{by De Morgan's laws}\\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{by the double negative law}\\
&\equiv (p \vee (\sim q \wedge q) && \text{by the distributive law}\\
&\equiv p \vee (q \wedge \sim q) && \text{by the commutative law for } \wedge\\
&\equiv p \vee \mathbf{c} && \text{by the negation law}\\
&\equiv p && \text{by the identity law} \quad \blacksquare
\end{aligned}$$

Skill in simplifying statement forms is useful in constructing logically efficient computer programs and in designing digital logic circuits.

Although the properties in Theorem 2.1.1 can be used to prove the logical equivalence of two statement forms, they cannot be used to prove that statement forms are not logically equivalent. On the other hand, truth tables can always be used to determine both equivalence and nonequivalence, and truth tables are easy to program on a computer. When truth tables are used, however, checking for equivalence always requires $2^n$ steps, where $n$ is the number of variables. Sometimes you can quickly see that two statement forms are equivalent by Theorem 2.1.1, whereas it would take quite a bit of calculating to show their equivalence using truth tables. For instance, it follows immediately from the associative law for $\wedge$ that $p \wedge (\sim q \wedge \sim r) \equiv (p \wedge \sim q) \wedge \sim r$, whereas a truth table verification requires constructing a table with eight rows.

## TEST YOURSELF

Answers to Test Yourself questions are located at the end of each section.

1. An *and* statement is true when, and only when, both components are ______.
2. An *or* statement is false when, and only when, both components are ______.
3. Two statement forms are logically equivalent when, and only when, they always have ______.
4. De Morgan's laws say (1) that the negation of an *and* statement is logically equivalent to the ______ statement in which each component is ______, and (2) that the negation of an *or* statement is logically equivalent to the ______ statement in which each component is ______.
5. A tautology is a statement that is always ______.
6. A contradiction is a statement that is always ______.

## EXERCISE SET 2.1*

In each of 1–4 represent the common form of each argument using letters to stand for component sentences, and fill in the blanks so that the argument in part (b) has the same logical form as the argument in part (a).

**1.** **a.** If all integers are rational, then the number 1 is rational.
All integers are rational.
Therefore, the number 1 is rational.

**b.** If all algebraic expressions can be written in prefix notation, then ______________________.
__________________________________.
Therefore, $(a+2b)(a^2-b)$ can be written in prefix notation.

**2.** **a.** If all computer programs contain errors, then this program contains an error.
This program does not contain an error.
Therefore, it is not the case that all computer programs contain errors.

**b.** If _______, then _______.
2 is not odd.
Therefore, it is not the case that all prime numbers are odd.

**3.** **a.** This number is even or this number is odd.
This number is not even.
Therefore, this number is odd.

**b.** _______ or logic is confusing.
My mind is not shot.
Therefore, _______.

**4.** **a.** If the program syntax is faulty, then the computer will generate an error message.
If the computer generates an error message, then the program will not run.
Therefore, if the program syntax is faulty, then the program will not run.

**b.** If this simple graph _______, then it is complete.
If this graph _______, then any two of its vertices can be joined by a path.
Therefore, if this simple graph has 4 vertices and 6 edges, then _______.

**5.** Indicate which of the following sentences are statements.

**a.** 1,024 is the smallest four-digit number that is a perfect square.

**b.** She is a mathematics major.

**c.** $128 = 2^6$

**d.** $x = 2^6$

Write the statements in 6–9 in symbolic form using the symbols $\sim$, $\vee$, and $\wedge$ and the indicated letters to represent component statements.

**6.** Let $s$ = "stocks are increasing" and $i$ = "interest rates are steady."

**a.** Stocks are increasing but interest rates are steady.

**b.** Neither are stocks increasing nor are interest rates steady.

**7.** Juan is a math major but not a computer science major. ($m$ = "Juan is a math major," $c$ = "Juan is a computer science major")

**8.** Let $h$ = "John is healthy," $w$ = "John is wealthy," and $s$ = "John is wise."

**a.** John is healthy and wealthy but not wise.

**b.** John is not wealthy but he is healthy and wise.

**c.** John is neither healthy, wealthy, nor wise.

**d.** John is neither wealthy nor wise, but he is healthy.

**e.** John is wealthy, but he is not both healthy and wise.

**9.** Let $p$ = "$x > 5$," $q$ = "$x = 5$," and $r$ = "$10 > x$."

**a.** $x \geq 5$

**b.** $10 > x > 5$

**c.** $10 > x \geq 5$

**10.** Let $p$ be the statement "DATAENDFLAG is off," $q$ the statement "ERROR equals 0," and $r$ the statement "SUM is less than 1,000." Express the following sentences in symbolic notation.

**a.** DATAENDFLAG is off, ERROR equals 0, and SUM is less than 1,000.

**b.** DATAENDFLAG is off but ERROR is not equal to 0.

**c.** DATAENDFLAG is off; however, ERROR is not 0 or SUM is greater than or equal to 1,000.

**d.** DATAENDFLAG is on and ERROR equals 0 but SUM is greater than or equal to 1,000.

**e.** Either DATAENDFLAG is on or it is the case that both ERROR equals 0 and SUM is less than 1,000.

**11.** In the following sentence, is the word *or* used in its inclusive or exclusive sense? A team wins the playoffs if it wins two games in a row or a total of three games.

---

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol * signals that an exercise is more challenging than usual.

Write truth tables for the statement forms in 12–15.

**12.** $\sim p \wedge q$

**13.** $\sim(p \wedge q) \vee (p \vee q)$

**14.** $p \wedge (q \wedge r)$

**15.** $p \wedge (\sim q \vee r)$

Determine whether the statement forms in 16–24 are logically equivalent. In each case, construct a truth table and include a sentence justifying your answer. Your sentence should show that you understand the meaning of logical equivalence.

**16.** $p \vee (p \wedge q)$ and $p$

**17.** $\sim(p \wedge q)$ and $\sim p \wedge \sim q$

**18.** $p \vee \mathbf{t}$ and $\mathbf{t}$

**19.** $p \wedge \mathbf{t}$ and $p$

**20.** $p \wedge \mathbf{c}$ and $p \vee \mathbf{c}$

**21.** $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$

**22.** $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$

**23.** $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$

**24.** $(p \vee q) \vee (p \wedge r)$ and $(p \vee q) \wedge r$

Use De Morgan's laws to write negations for the statements in 25–30.

**25.** Hal is a math major and Hal's sister is a computer science major.

**26.** Sam is an orange belt and Kate is a red belt.

**27.** The connector is loose or the machine is unplugged.

**28.** The train is late or my watch is fast.

**29.** This computer program has a logical error in the first ten lines or it is being run with an incomplete data set.

**30.** The dollar is at an all-time high and the stock market is at a record low.

**31.** Let $s$ be a string of length 2 with characters from $\{0, 1, 2\}$, and define statements $a$, $b$, $c$, and $d$ as follows:

$a$ = "the first character of $s$ is 0"

$b$ = "the first character of $s$ is 1"

$c$ = "the second character of $s$ is 1"

$d$ = "the second character of $s$ is 2".

Describe the set of all strings for which each of the following is true.

**a.** $(a \vee b) \wedge (c \vee d)$

**b.** $(\sim(a \vee b)) \wedge (c \vee d)$

**c.** $((\sim a) \vee b) \wedge (c \vee (\sim d))$

Assume $x$ is a particular real number and use De Morgan's laws to write negations for the statements in 32–37.

**32.** $-2 < x < 7$

**33.** $-10 < x < 2$

**34.** $x < 2$ or $x > 5$

**35.** $x \leq -1$ or $x > 1$

**36.** $1 > x \geq -3$

**37.** $0 > x \geq -7$

In 38 and 39, imagine that *num_orders* and *num_instock* are particular values, such as might occur during execution of a computer program. Write negations for the following statements.

**38.** (*num_orders* > 100 and *num_instock* ≤ 500) or *num_instock* < 200

**39.** (*num_orders* < 50 and *num_instock* > 300) or (50 ≤ *num_orders* < 75 and *num_instock* > 500)

Use truth tables to establish which of the statement forms in 40–43 are tautologies and which are contradictions.

**40.** $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$

**41.** $(p \wedge \sim q) \wedge (\sim p \vee q)$

**42.** $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$

**43.** $(\sim p \vee q) \vee (p \wedge \sim q)$

**44.** Recall that $a < x < b$ means that $a < x$ and $x < b$. Also $a \leq b$ means that $a < b$ or $a = b$. Find all real numbers that satisfy the following inequalities.

**a.** $2 < x \leq 0$

**b.** $1 \leq x < -1$

**45.** Determine whether the statements in (a) and (b) are logically equivalent.

**a.** Bob is both a math and computer science major and Ann is a math major, but Ann is not both a math and computer science major.

**b.** It is not the case that both Bob and Ann are both math and computer science majors, but it is the case that Ann is a math major and Bob is both a math and computer science major.

*__46.__ Let the symbol $\oplus$ denote *exclusive or*; so $p \oplus q \equiv (p \vee q) \wedge \sim(p \wedge q)$. Hence the truth table for $p \oplus q$ is as follows:

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

**a.** Find simpler statement forms that are logically equivalent to $p \oplus p$ and $(p \oplus p) \oplus p$.

**b.** Is $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$? Justify your answer.

**c.** Is $(p \oplus q) \wedge r \equiv (p \wedge r) \oplus (q \wedge r)$? Justify your answer.

*__47.__ In logic and in standard English, a double negative is equivalent to a positive. There is one fairly common English usage in which a "double positive" is equivalent to a negative. What is it? Can you think of others?

**In 48 and 49 below, a logical equivalence is derived from Theorem 2.1.1. Supply a reason for each step.**

**48.** $(p \wedge \sim q) \vee (p \wedge q) \equiv p \wedge (\sim q \vee q)$ by (a)
$\equiv p \wedge (q \vee \sim q)$ by (b)
$\equiv p \wedge \mathbf{t}$ by (c)
$\equiv p$ by (d)

Therefore, $(p \wedge \sim q) \vee (p \wedge q) \equiv p$.

**49.** $(p \vee \sim q) \wedge (\sim p \vee \sim q)$
$\equiv (\sim q \vee p) \wedge (\sim q \vee \sim p)$ by (a)
$\equiv \sim q \vee (p \wedge \sim p)$ by (b)
$\equiv \sim q \vee \mathbf{c}$ by (c)
$\equiv \sim q$ by (d)

Therefore, $(p \vee \sim q) \wedge (\sim p \vee \sim q) \equiv \sim q$.

**Use Theorem 2.1.1 to verify the logical equivalences in 50–54. Supply a reason for each step.**

**50.** $(p \wedge \sim q) \vee p \equiv p$

**51.** $p \wedge (\sim q \vee p) \equiv p$

**52.** $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$

**53.** $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

**54.** $(p \wedge (\sim(\sim p \vee q))) \vee (p \wedge q) \equiv p$

## ANSWERS FOR TEST YOURSELF

1. true 2. false 3. the same truth values 4. *or*; negated; *and*; negated 5. true 6. false

# 2.2 Conditional Statements

> *. . . hypothetical reasoning implies the subordination of the real to the realm of the possible . . .* —Jean Piaget, 1972

When you make a logical inference or deduction, you reason *from* a hypothesis *to* a conclusion. Your aim is to be able to say, "*If* such and such is known, *then* something or other must be the case."

Let $p$ and $q$ be statements. A sentence of the form "If $p$ then $q$" is denoted symbolically by "$p \to q$"; $p$ is called the *hypothesis* and $q$ is called the *conclusion*. For instance, consider the following statement:

$$\text{If }\underbrace{\text{4,686 is divisible by 6}}_{\text{hypothesis}}\text{, then }\underbrace{\text{4,686 is divisible by 3}}_{\text{conclusion}}$$

Such a sentence is called *conditional* because the truth of statement $q$ is conditioned on the truth of statement $p$.

The notation $p \to q$ indicates that $\to$ is a connective, like $\wedge$ or $\vee$, which can be used to join statements to create new statements. To define $p \to q$ as a statement, therefore, we must specify the truth values for $p \to q$ as we specified truth values for $p \wedge q$ and for $p \vee q$. As is the case with the other connectives, the formal definition of truth values for $\to$ (if-then) is based on its everyday, intuitive meaning. Consider an example.

> Suppose you go to interview for a job at a store and the owner of the store makes you the following promise:
>
> > If you show up for work Monday morning, then you will get the job.
>
> Under what circumstances are you justified in saying the owner spoke falsely? That is, under what circumstances is the above sentence false? The answer is: You *do* show up for work Monday morning and you do *not* get the job.
>
> After all, the owner's promise only says you will get the job *if* a certain condition (showing up for work Monday morning) is met; it says nothing about what will happen if the condition is *not* met. So if the condition is not met, you cannot in fairness say the promise is false regardless of whether or not you get the job.

The above example was intended to convince you that *the only combination of circumstances in which you would call a conditional sentence false occurs when the hypothesis is true and the conclusion is false*. In all other cases, you would not call the sentence false. This implies that the only row of the truth table for $p \longrightarrow q$ that should be filled in with an F is the row where $p$ is T and $q$ is F. No other row should contain an F. But each row of a truth table must be filled in with either a T or an F. Thus all other rows of the truth table for $p \longrightarrow q$ must be filled in with T's.

Truth Table for $p \longrightarrow q$

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**Definition**

If $p$ and $q$ are statement variables, the **conditional** of $q$ by $p$ is "If $p$ then $q$" or "$p$ implies $q$" and is denoted $p \longrightarrow q$. It is false when $p$ is true and $q$ is false; otherwise it is true. We call $p$ the **hypothesis** (or **antecedent**) of the conditional and $q$ the **conclusion** (or **consequent**).

A conditional statement that is true by virtue of the fact that its hypothesis is false is often called **vacuously true** or **true by default**. Thus the statement "If you show up for work Monday morning, then you will get the job" is vacuously true if you do not show up for work Monday morning. In general, when the "if" part of an if-then statement is false, the statement as a whole is said to be true, regardless of whether the conclusion is true or false.

### Example 2.2.1 A Conditional Statement with a False Hypothesis

Consider the statement

$$\text{If } 0 = 1 \text{ then } 1 = 2.$$

As strange as it may seem, since the hypothesis of this statement is false, the statement as a whole is true. ■

**Note** For example, if you hypothesize that $0 = 1$, then, by adding 1 to both sides of the equation, you can deduce that $1 = 2$.

The philosopher Willard Van Orman Quine advises against using the phrase "$p$ implies $q$" to mean "$p \to q$" because the word *implies* suggests that $q$ can be logically deduced from $p$ and this is often not the case. Nonetheless, the phrase is used by many people, probably because it is a convenient replacement for the $\to$ symbol. And, of course, in many cases a conclusion can be deduced from a hypothesis, even when the hypothesis is false.

In expressions that include $\to$ as well as other logical operators such as $\wedge$, $\vee$, and $\sim$, the **order of operations** is that $\to$ is performed last. Thus, according to the specification of order of operations in Section 2.1, $\sim$ is performed first, then $\wedge$ and $\vee$, and finally $\to$.

**Example 2.2.2**

### Truth Table for $p \vee \sim q \to \sim p$

Construct a truth table for the statement form $p \vee \sim q \to \sim p$.

**Solution** By the order of operations given above, the following two expressions are equivalent: $p \vee \sim q \to \sim p$ and $(p \vee (\sim q)) \to (\sim p)$, and this order governs the construction of the truth table. First fill in the four possible combinations of truth values for $p$ and $q$, and then enter the truth values for $\sim p$ and $\sim q$ using the definition of negation. Next fill in the $p \vee \sim q$ column using the definition of $\vee$. Finally, fill in the $p \vee \sim q \to \sim p$ column using the definition of $\to$.

**Note** The only rows in which the hypothesis $p \vee \sim q$ is true and the conclusion $\sim p$ is false are the first and second rows. So you put F's in those two rows and T's in the other two rows.

| | | conclusion | | hypothesis | |
|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \vee \sim q$ | $p \vee \sim q \to \sim p$ |
| T | T | F | F | T | F |
| T | F | F | T | T | F |
| F | T | T | F | F | T |
| F | F | T | T | T | T |

■

## Logical Equivalences Involving $\to$

Imagine that you are trying to solve a problem involving three statements: $p$, $q$, and $r$. Suppose you know that the truth of $r$ follows from the truth of $p$ *and also* that the truth of $r$ follows from the truth of $q$. Then no matter whether $p$ or $q$ is the case, the truth of $r$ must follow. The division-into-cases method of analysis is based on this idea.

**Example 2.2.3**

### Division into Cases: Showing That $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$

Use truth tables to show the logical equivalence of the statement forms $p \vee q \to r$ and $(p \to r) \wedge (q \to r)$. Annotate the table with a sentence of explanation.

**Solution** First fill in the eight possible combinations of truth values for $p$, $q$, and $r$. Then fill in the columns for $p \vee q$, $p \to r$, and $q \to r$ using the definitions of *or* and *if-then*. For instance, the $p \to r$ column has F's in the second and fourth rows because these are the rows in which $p$ is true and $r$ is false. Next fill in the $p \vee q \to r$ column using the definition of *if-then*. The rows in which the hypothesis $p \vee q$ is true and the conclusion $r$ is false are the second, fourth, and sixth. So F's go in these rows and T's in all the others. The complete table shows that $p \vee q \to r$ and $(p \to r) \wedge (q \to r)$ have the same truth values for each combination of truth values of $p$, $q$, and $r$. Hence the two statement forms are logically equivalent.

| $p$ | $q$ | $r$ | $p \vee q$ | $p \to r$ | $q \to r$ | $p \vee q \to r$ | $(p \to r) \wedge (q \to r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | F |
| T | F | T | T | T | T | T | T |
| T | F | F | T | F | T | F | F |
| F | T | T | T | T | T | T | T |
| F | T | F | T | T | F | F | F |
| F | F | T | F | T | T | T | T |
| F | F | F | F | T | T | T | T |

$p \vee q \to r$ and $(p \to r) \wedge (q \to r)$ always have the same truth values, so they are logically equivalent ■

## *Representation of If-Then as Or*

In exercise 13(a) at the end of this section you are asked to use truth tables to show that

$$p \to q \equiv \sim p \vee q.$$

The logical equivalence of "if $p$ then $q$" and "not $p$ or $q$" is occasionally used in everyday speech. Here is one instance.

### **Example 2.2.4** Application of the Equivalence between $\sim p \vee q$ and $p \to q$

Rewrite the following statement in if-then form.

Either you get to work on time or you are fired.

**Solution** Let $\sim p$ be

You get to work on time.

and $q$ be

You are fired.

Then the given statement is $\sim p \vee q$. Also $p$ is

You do not get to work on time.

So the equivalent if-then version, $p \to q$, is

If you do not get to work on time, then you are fired. ■

## *The Negation of a Conditional Statement*

By definition, $p \to q$ is false if, and only if, its hypothesis, $p$, is true and its conclusion, $q$, is false. It follows that

The negation of "if $p$ then $q$" is logically equivalent to "$p$ and not $q$."

This can be restated symbolically as follows:

$$\boxed{\sim(p \to q) \equiv p \wedge \sim q}$$

To obtain this result you can also start from the logical equivalence $p \to q \equiv \sim p \vee q$. Take the negation of both sides to obtain

$$\begin{aligned}\sim(p \to q) &\equiv \sim(\sim p \vee q) \\ &\equiv \sim(\sim p) \wedge (\sim q) && \text{by De Morgan's laws} \\ &\equiv p \wedge \sim q && \text{by the double negative law.}\end{aligned}$$

Yet another way to derive this result is to construct truth tables for $\sim(p \to q)$ and for $p \wedge \sim q$ and to check that they have the same truth values. (See exercise 13(b) at the end of this section.)

**Example 2.2.5**

### Negations of *If-Then* Statements

Write negations for each of the following statements:

a. If my car is in the repair shop, then I cannot get to class.

b. If Sara lives in Athens, then she lives in Greece.

### Solution

a. My car is in the repair shop and I can get to class.

b. Sara lives in Athens and she does not live in Greece. (Sara might live in Athens, Georgia; Athens, Ohio; or Athens, Wisconsin.) ■

**Caution!** Remember that **the negation of an if-then statement does not start with the word *if*.**

It is tempting to write the negation of an if-then statement as another if-then statement. Please resist that temptation!

## *The Contrapositive of a Conditional Statement*

One of the most fundamental laws of logic is the equivalence between a conditional statement and its contrapositive.

**Definition**

The **contrapositive** of a conditional statement of the form "If $p$ then $q$" is

If $\sim q$ then $\sim p$.

Symbolically,

The contrapositive of $p \to q$ is $\sim q \to \sim p$.

The fact is that

A conditional statement is logically equivalent to its contrapositive.

You are asked to establish this equivalence in exercise 26 at the end of this section.

### Example 2.2.6 Writing the Contrapositive

Write each of the following statements in its equivalent contrapositive form:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

### Solution

a. If Howard cannot swim to the island, then Howard cannot swim across the lake.

b. If tomorrow is not Monday, then today is not Easter. ■

When you are trying to solve certain problems, you may find that the contrapositive form of a conditional statement is easier to work with than the original statement. Replacing a statement by its contrapositive may give the extra push that helps you over the top in your search for a solution. This logical equivalence is also the basis for one of the most important laws of deduction, modus tollens (to be explained in Section 2.3), and for the contrapositive method of proof (to be explained in Section 4.7).

## The Converse and Inverse of a Conditional Statement

The fact that a conditional statement and its contrapositive are logically equivalent is very important and has wide application. Two other variants of a conditional statement are *not* logically equivalent to the statement.

### Definition

Suppose a conditional statement of the form "If $p$ then $q$" is given.

1. The **converse** is "If $q$ then $p$."
2. The **inverse** is "If $\sim p$ then $\sim q$."

Symbolically,

$$\text{The converse of } p \rightarrow q \text{ is } q \rightarrow p,$$

and

$$\text{The inverse of } p \rightarrow q \text{ is } \sim p \rightarrow \sim q.$$

### Example 2.2.7 Writing the Converse and the Inverse

Write the converse and inverse of each of the following statements:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

### Solution

a. *Converse:* If Howard can swim to the island, then Howard can swim across the lake.

*Inverse:* If Howard cannot swim across the lake, then Howard cannot swim to the island.

b. *Converse:* If tomorrow is Monday, then today is Easter.

*Inverse:* If today is not Easter, then tomorrow is not Monday. ■

**Caution!** Many people believe that if a conditional statement is true, then its converse and inverse must also be true. This is not correct! The converse might be true, but it does not have to be true.

Note that while the statement “If today is Easter, then tomorrow is Monday” is always true, both its converse and inverse are false on every Sunday except Easter.

1. A conditional statement and its converse are *not* logically equivalent.
2. A conditional statement and its inverse are *not* logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

In exercises 24, 25, and 27 at the end of this section, you are asked to use truth tables to verify the statements in the box above. Note that the truth of statement 3 also follows from the observation that the inverse of a conditional statement is the contrapositive of its converse.

## *Only If and the Biconditional*

To say “$p$ only if $q$” means that $p$ can take place *only* if $q$ takes place also. That is, if $q$ does not take place, then $p$ cannot take place. Another way to say this is that if $p$ occurs, then $q$ must also occur (by the logical equivalence between a statement and its contrapositive).

**Definition**

If $p$ and $q$ are statements,

$p$ **only if** $q$ means “if not $q$ then not $p$,”

or, equivalently,

“if $p$ then $q$.”

**Example 2.2.8**

### Converting *Only If* to *If-Then*

Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other.

> John will break the world’s record for the mile run only if he runs the mile in under four minutes.

**Solution** *Version 1:* If John does not run the mile in under four minutes, then he will not break the world’s record.

*Version 2:* If John breaks the world’s record, then he will have run the mile in under four minutes. ■

**Caution!** “$p$ only if $q$” does *not* mean “$p$ if $q$.”

Note that it is possible for “$p$ only if $q$” to be true at the same time that “$p$ if $q$” is false. For instance, to say that John will break the world’s record only if he runs the mile in under four minutes does not mean that John will break the world’s record if he runs the mile in under four minutes. His time could be under four minutes but still not be fast enough to break the record.

**Definition**

Given statement variables $p$ and $q$, the **biconditional of $p$ and $q$** is "$p$ if, and only if, $q$" and is denoted $p \leftrightarrow q$. It is true if both $p$ and $q$ have the same truth values and is false if $p$ and $q$ have opposite truth values. The words *if and only if* are sometimes abbreviated **iff**.

The biconditional has the following truth table:

Truth Table for $p \leftrightarrow q$

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

In order of operations $\leftrightarrow$ is coequal with $\rightarrow$. As with $\wedge$ and $\vee$, the only way to indicate precedence between them is to use parentheses. The full hierarchy of operations for the five logical operators is shown below.

### Order of Operations for Logical Operators

1. $\sim$ Evaluate negations first.
2. $\wedge, \vee$ Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3. $\rightarrow, \leftrightarrow$ Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

According to the separate definitions of *if* and *only if*, saying "$p$ if, and only if, $q$" should mean the same as saying both "$p$ if $q$" and "$p$ only if $q$." The following annotated truth table shows that this is the case:

Truth Table Showing That $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

| $p$ | $q$ | $p \rightarrow q$ | $q \rightarrow p$ | $p \leftrightarrow q$ | $(p \rightarrow q) \wedge (q \rightarrow p)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |

$p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$
always have the same truth values,
so they are logically equivalent

**Example 2.2.9** ***If* and *Only If***

Rewrite the following statement as a conjunction of two if-then statements:

> This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

**Solution** If this program is correct, then it produces the correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct. ■

## *Necessary and Sufficient Conditions*

The phrases *necessary condition* and *sufficient condition*, as used in formal English, correspond exactly to their definitions in logic.

**Definition**

If $r$ and $s$ are statements:

|  |  |  |
|---|---|---|
| $r$ is a **sufficient condition** for $s$ | means | "if $r$ then $s$." |
| $r$ is a **necessary condition** for $s$ | means | "if not $r$ then not $s$." |

In other words, to say "$r$ is a sufficient condition for $s$" means that the occurrence of $r$ is *sufficient* to guarantee the occurrence of $s$. On the other hand, to say "$r$ is a necessary condition for $s$" means that if $r$ does not occur, then $s$ cannot occur either:

> The occurrence of $r$ is *necessary* to obtain the occurrence of $s$. Note that because of the equivalence between a statement and its contrapositive,

$r$ is a necessary condition for $s$ also means "if $s$ then $r$."

Consequently,

$r$ is a necessary and sufficient condition for $s$ means "$r$ if, and only if, $s$."

**Example 2.2.10** **Interpreting Necessary and Sufficient Conditions**

Consider the statement "If John is eligible to vote, then he is at least 18 years old." The truth of the condition "John is eligible to vote" is *sufficient* to ensure the truth of the condition "John is at least 18 years old." In addition, the condition "John is at least 18 years old" is *necessary* for the condition "John is eligible to vote" to be true. If John were younger than 18, then he would not be eligible to vote. ■

**Example 2.2.11** **Converting a Sufficient Condition to If-Then Form**

Rewrite the following statement in the form "If $A$ then $B$":

> Pia's birth on U.S. soil is a sufficient condition
> for her to be a U.S. citizen.

**Solution** If Pia was born on U.S. soil, then she is a U.S. citizen. ■

### **Example 2.2.12** Converting a Necessary Condition to If-Then Form

Use the contrapositive to rewrite the following statement in two ways:

> George's attaining age 35 is a necessary condition for his being president of the United States.

**Solution** *Version 1:* If George has not attained the age of 35, then he cannot be president of the United States.

*Version 2:* If George can be president of the United States, then he has attained the age of 35. ■

## *Remarks*

1. *In logic, a hypothesis and conclusion are not required to have related subject matters.*

   In ordinary speech we never say things like "If computers are machines, then Babe Ruth was a baseball player" or "If $2 + 2 = 5$, then Mickey Mouse is president of the United States." We formulate a sentence like "If $p$ then $q$" only if there is some connection of content between $p$ and $q$.

   In logic, however, the two parts of a conditional statement need not have related meanings. The reason? If there were such a requirement, who would enforce it? What one person perceives as two unrelated clauses may seem related to someone else. There would have to be a central arbiter to check each conditional sentence before anyone could use it, to be sure its clauses were in proper relation. This is impractical, to say the least!

   Thus a statement like "if computers are machines, then Babe Ruth was a baseball player" is allowed, and it is even called true because both its hypothesis and its conclusion are true. Similarly, the statement "If $2 + 2 = 5$, then Mickey Mouse is president of the United States" is allowed and is called true because its hypothesis is false, even though doing so may seem ridiculous.

   In mathematics it often happens that a carefully formulated definition that successfully covers the situations for which it was primarily intended is later seen to be satisfied by some extreme cases that the formulator did not have in mind. But those are the breaks, and it is important to get into the habit of exploring definitions fully to seek out and understand *all* their instances, even the unusual ones.

2. *In informal language, simple conditionals are often used to mean biconditionals.*

   The formal statement "$p$ if, and only if, $q$" is seldom used in ordinary language. Frequently, when people intend the biconditional they leave out either the *and only if* or the *if and*. That is, they say either "$p$ if $q$" or "$p$ only if $q$" when they really mean "$p$ if, and only if, $q$." For example, consider the statement "You will get dessert if, and only if, you eat your dinner." Logically, this is equivalent to the conjunction of the following two statements.

   *Statement 1:* If you eat your dinner, then you will get dessert.
   *Statement 2:* You will get dessert only if you eat your dinner.
   or
   If you do not eat your dinner, then you will not get dessert.

   Now how many parents in the history of the world have said to their children "You will get dessert if, and only if, you eat your dinner"? Not many! Most say either "If you eat your dinner, you will get dessert" (these take the positive approach—they emphasize the reward) or "You will get dessert only if you eat your dinner" (these take the negative approach—they emphasize the punishment). Yet the parents who promise the reward intend to suggest the punishment as well, and those who threaten the punishment will certainly give the reward if it is earned. Both sets of parents expect that their conditional statements will be interpreted as biconditionals.

Since we often (correctly) interpret conditional statements as biconditionals, it is not surprising that we may come to believe (mistakenly) that conditional statements are always logically equivalent to their inverses and converses. In formal settings, however, statements must have unambiguous interpretations. If-then statements can't sometimes mean "if-then" and other times mean "if and only if." When using language in mathematics, science, or other situations where precision is important, it is essential to interpret if-then statements according to the formal definition and not to confuse them with their converses and inverses.

## TEST YOURSELF

1. An *if-then* statement is false if, and only if, the hypothesis is ________ and the conclusion is ________.
2. The negation of "if $p$ then $q$" is ________.
3. The converse of "if $p$ then $q$" is ________.
4. The contrapositive of "if $p$ then $q$" is ________.
5. The inverse of "if $p$ then $q$" is ________.
6. A conditional statement and its contrapositive are ________.
7. A conditional statement and its converse are not ________.
8. "$R$ is a sufficient condition for $S$" means "if ________ then ________."
9. "$R$ is a necessary condition for $S$" means "if ________ then ________."
10. "$R$ only if $S$" means "if ________ then ________."

## EXERCISE SET 2.2

**Rewrite the statements in 1–4 in if-then form.**

1. This loop will repeat exactly $N$ times if it does not contain a **stop** or a **go to**.
2. I am on time for work if I catch the 8:05 bus.
3. Freeze or I'll shoot.
4. Fix my ceiling or I won't pay my rent.

**Construct truth tables for the statement forms in 5–11.**

5. $\sim p \vee q \rightarrow \sim q$
6. $(p \vee q) \vee (\sim p \wedge q) \rightarrow q$
7. $p \wedge \sim q \rightarrow r$
8. $\sim p \vee q \rightarrow r$
9. $p \wedge \sim r \leftrightarrow q \vee r$
10. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$
11. $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$
12. Use the logical equivalence established in Example 2.2.3, $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$, to rewrite the following statement. (Assume that $x$ represents a fixed real number.)

$$\text{If } x > 2 \text{ or } x < -2, \text{ then } x^2 > 4.$$

13. Use truth tables to verify the following logical equivalences. Include a few words of explanation with your answers.
    a. $p \rightarrow q \equiv \sim p \vee q$
    b. $\sim(p \rightarrow q) \equiv p \wedge \sim q$.

*H* 14. a. Show that the following statement forms are all logically equivalent:

$$p \rightarrow q \vee r, \quad p \wedge \sim q \rightarrow r, \quad \text{and} \quad p \wedge \sim r \rightarrow q$$

b. Use the logical equivalences established in part (a) to rewrite the following sentence in two different ways. (Assume that $n$ represents a fixed integer.)

If $n$ is prime, then $n$ is odd or $n$ is 2.

**15.** Determine whether the following statement forms are logically equivalent:

$$p \to (q \to r) \quad \text{and} \quad (p \to q) \to r$$

**In 16 and 17, write each of the two statements in symbolic form and determine whether they are logically equivalent. Include a truth table and a few words of explanation to show that you understand what it means for statements to be logically equivalent.**

**16.** If you paid full price, you didn't buy it at Crown Books. You didn't buy it at Crown Books or you paid full price.

**17.** If 2 is a factor of $n$ and 3 is a factor of $n$, then 6 is a factor of $n$. 2 is not a factor of $n$ or 3 is not a factor of $n$ or 6 is a factor of $n$.

**18.** Write each of the following three statements in symbolic form and determine which pairs are logically equivalent. Include truth tables and a few words of explanation.

If it walks like a duck and it talks like a duck, then it is a duck.

Either it does not walk like a duck or it does not talk like a duck, or it is a duck.

If it does not walk like a duck and it does not talk like a duck, then it is not a duck.

**19.** True or false? The negation of "If Sue is Luiz's mother, then Ali is his cousin" is "If Sue is Luiz's mother, then Ali is not his cousin."

**20.** Write negations for each of the following statements. (Assume that all variables represent fixed quantities or entities, as appropriate.)
- **a.** If $P$ is a square, then $P$ is a rectangle.
- **b.** If today is New Year's Eve, then tomorrow is January.
- **c.** If the decimal expansion of $r$ is terminating, then $r$ is rational.
- **d.** If $n$ is prime, then $n$ is odd or $n$ is 2.
- **e.** If $x$ is nonnegative, then $x$ is positive or $x$ is 0.
- **f.** If Tom is Ann's father, then Jim is her uncle and Sue is her aunt.
- **g.** If $n$ is divisible by 6, then $n$ is divisible by 2 and $n$ is divisible by 3.

**21.** Suppose that $p$ and $q$ are statements so that $p \to q$ is false. Find the truth values of each of the following:

**a.** $\sim p \to q$ **b.** $p \vee q$ **c.** $q \to p$

***H* 22.** Write contrapositives for the statements of exercise 20.

***H* 23.** Write the converse and inverse for each statement of exercise 20.

**Use truth tables to establish the truth of each statement in 24–27.**

**24.** A conditional statement is not logically equivalent to its converse.

**25.** A conditional statement is not logically equivalent to its inverse.

**26.** A conditional statement and its contrapositive are logically equivalent to each other.

**27.** The converse and inverse of a conditional statement are logically equivalent to each other.

***H* 28.** "Do you mean that you think you can find out the answer to it?" said the March Hare.

> "Exactly so," said Alice.
>
> "Then you should say what you mean," the March Hare went on.
>
> "I do," Alice hastily replied; "at least—at least I mean what I say—that's the same thing, you know."
>
> "Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
>
> —from "A Mad Tea-Party" in *Alice in Wonderland*, by Lewis Carroll

The Hatter is right. "I say what I mean" is not the same thing as "I mean what I say." Rewrite each of these two sentences in if-then form and explain the logical relation between them. (This exercise is referred to in the introduction to Chapter 4.)

**If statement forms $P$ and $Q$ are logically equivalent, then $P \leftrightarrow Q$ is a tautology. Conversely, if $P \leftrightarrow Q$ is a tautology, then $P$ and $Q$ are logically equivalent. Use $\leftrightarrow$ to convert each of the logical equivalences in 29–31 to a tautology. Then use a truth table to verify each tautology.**

**29.** $p \to (q \vee r) \equiv (p \wedge \sim q) \to r$

**30.** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

**31.** $p \to (q \to r) \equiv (p \wedge q) \to r$

**Rewrite each of the statements in 32 and 33 as a conjunction of two if-then statements.**

**32.** This quadratic equation has two distinct real roots if, and only if, its discriminant is greater than zero.

**33.** This integer is even if, and only if, it equals twice some integer.

**Rewrite the statements in 34 and 35 in if-then form in two ways, one of which is the contrapositive of the other. Use the formal definition of "only if."**

**34.** The Cubs will win the pennant only if they win tomorrow's game.

**35.** Sam will be allowed on Signe's racing boat only if he is an expert sailor.

**36.** Taking the long view on your education, you go to the Prestige Corporation and ask what you should do in college to be hired when you graduate. The personnel director replies that you will be hired *only if* you major in mathematics or computer science, get a B average or better, and take accounting. You do, in fact, become a math major, get a B+ average, and take accounting. You return to Prestige Corporation, make a formal application, and are turned down. Did the personnel director lie to you?

**Some programming languages use statements of the form "*r* unless *s*" to mean that as long as *s* does not happen, then *r* will happen. More formally:**

> **Definition:** If $r$ and $s$ are statements,
>
> $$\textbf{r unless s} \quad \text{means} \quad \text{if } {\sim}s \text{ then } r.$$

**In 37–39, rewrite the statements in if-then form.**

**37.** Payment will be made on fifth unless a new hearing is granted.

**38.** Ann will go unless it rains.

**39.** This door will not open unless a security code is entered.

**Rewrite the statements in 40 and 41 in if-then form.**

**40.** Catching the 8:05 bus is a sufficient condition for my being on time for work.

**41.** Having two 45° angles is a sufficient condition for this triangle to be a right triangle.

**Use the contrapositive to rewrite the statements in 42 and 43 in if-then form in two ways.**

**42.** Being divisible by 3 is a necessary condition for this number to be divisible by 9.

**43.** Doing homework regularly is a necessary condition for Jim to pass the course.

**Note that "a sufficient condition for *s* is *r*" means *r* is a sufficient condition for *s* and that "a necessary condition for *s* is *r*" means *r* is a necessary condition for *s*. Rewrite the statements in 44 and 45 in if-then form.**

**44.** A sufficient condition for Jon's team to win the championship is that it win the rest of its games.

**45.** A necessary condition for this computer program to be correct is that it not produce error messages during translation.

**46.** "If compound $X$ is boiling, then its temperature must be at least 150°C." Assuming that this statement is true, which of the following must also be true?

- **a.** If the temperature of compound $X$ is at least 150°C, then compound $X$ is boiling.
- **b.** If the temperature of compound $X$ is less than 150°C, then compound $X$ is not boiling.
- **c.** Compound $X$ will boil only if its temperature is at least 150°C.
- **d.** If compound $X$ is not boiling, then its temperature is less than 150°C.
- **e.** A necessary condition for compound $X$ to boil is that its temperature be at least 150°C.
- **f.** A sufficient condition for compound $X$ to boil is that its temperature be at least 150°C.

**In 47–50 (a) use the logical equivalences $p \to q \equiv {\sim}p \vee q$ and $p \leftrightarrow q \equiv ({\sim}p \vee q) \wedge ({\sim}q \vee p)$ to rewrite the given statement forms without using the symbol $\to$ or $\leftrightarrow$, and (b) use the logical equivalence $p \vee q \equiv {\sim}({\sim}p \wedge {\sim}q)$ to rewrite each statement form using only $\wedge$ and $\sim$.**

**47.** $p \wedge {\sim}q \to r$

**48.** $p \vee {\sim}q \to r \vee q$

**49.** $(p \to r) \leftrightarrow (q \to r)$

**50.** $(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$

**51.** Given any statement form, is it possible to find a logically equivalent form that uses only $\sim$ and $\wedge$? Justify your answer.

## ANSWERS FOR TEST YOURSELF

**1.** true; false **2.** $p \wedge \sim q$ **3.** if $q$ then $p$ **4.** if $\sim q$ then $\sim p$ **5.** if $\sim p$ then $\sim q$ **6.** logically equivalent **7.** logically equivalent **8.** $R$; $S$ **9.** $S$; $R$ **10.** $R$; $S$

# 2.3 Valid and Invalid Arguments

*"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."* —Lewis Carroll, *Through the Looking Glass*

In mathematics and logic an argument is not a dispute. It is simply a sequence of statements ending in a conclusion. In this section we show how to determine whether an argument is valid—that is, whether the conclusion follows *necessarily* from the preceding statements. We will show that this determination depends only on the form of an argument, not on its content.

It was shown in Section 2.1 that the logical form of an argument can be abstracted from its content. For example, the argument

If Socrates is a man, then Socrates is mortal.
Socrates is a man.
∴ Socrates is mortal.

has the abstract form

If $p$ then $q$
$p$
$\therefore q$

When considering the abstract form of an argument, think of $p$ and $q$ as variables for which statements may be substituted. An argument form is called *valid* if, and only if, whenever statements are substituted that make all the premises true, the conclusion is also true.

**Definition**

An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion**. The symbol ∴, which is read "therefore," is normally placed just before the conclusion.

To say that an *argument form* is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an *argument* is **valid** means that its form is valid.

The crucial fact about a valid argument is that the truth of its conclusion follows *necessarily* or *inescapably* or *by logical form alone* from the truth of its premises. It is impossible to have a valid argument with all true premises and a false conclusion. When an argument is valid and its premises are true, the truth of the conclusion is said to be *inferred* or *deduced* from the truth of the premises. If a conclusion "ain't necessarily so," then it isn't a valid deduction.

### Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row**. If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid. If the conclusion in *every* critical row is true, then the argument form is valid.

**Example 2.3.1**

### Determining Validity or Invalidity

Determine whether the following argument form is valid or invalid by drawing a truth table, indicating which columns represent the premises and which represent the conclusion, and annotating the table with a sentence of explanation. When you fill in the table, you only need to indicate the truth values for the conclusion in the rows where all the premises are true (the critical rows) because the truth values of the conclusion in the other rows are irrelevant to the validity or invalidity of the argument.

$$p \to q \lor {\sim}r$$
$$q \to p \land r$$
$$\therefore\ p \to r$$

**Caution!** If at least one premise of an argument is false, then we have no information about the conclusion: It might be true or it might be false.

**Solution** The truth table shows that even though there are several situations in which the premises and the conclusion are all true (rows 1, 7, and 8), there is one situation (row 4) where the premises are true and the conclusion is false.

| | | | | | | premises | | conclusion |
|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | ${\sim}r$ | $q \lor {\sim}r$ | $p \land r$ | $p \to q \lor {\sim}r$ | $q \to p \land r$ | $p \to r$ |
| T | T | T | F | T | T | T | T | T |
| T | T | F | T | T | F | T | F | |
| T | F | T | F | F | T | F | T | |
| T | F | F | T | T | F | T | T | F |
| F | T | T | F | T | F | T | F | |
| F | T | F | T | T | F | T | F | |
| F | F | T | F | F | F | T | T | T |
| F | F | F | T | T | F | T | T | T |

This row shows that an argument of this form can have true premises and a false conclusion. Hence this form of argument is invalid.

■

## *Modus Ponens and Modus Tollens*

An argument form consisting of two premises and a conclusion is called a **syllogism**. The first and second premises are called the **major premise** and **minor premise**, respectively. The most famous form of syllogism in logic is called **modus ponens**. It has the following form:

If $p$ then $q$.
$p$
$\therefore q$

Here is an argument of this form:

If the sum of the digits of 371,487 is divisible by 3,
then 371,487 is divisible by 3.
The sum of the digits of 371,487 is divisible by 3.
∴ 371,487 is divisible by 3.

The term *modus ponens* is Latin meaning "method of affirming" (the conclusion is an affirmation). Long before you saw your first truth table, you were undoubtedly being convinced by arguments of this form. Nevertheless, it is instructive to prove that modus ponens is a valid form of argument, if for no other reason than to confirm the agreement between the formal definition of validity and the intuitive concept. To do so, we construct a truth table for the premises and conclusion.

| | | premises | | conclusion | |
|---|---|---|---|---|---|
| $p$ | $q$ | $p \to q$ | $p$ | $q$ | |
| T | T | T | T | T | ← critical row |
| T | F | F | T | | |
| F | T | T | F | | |
| F | F | T | F | | |

The first row is the only one in which both premises are true, and the conclusion in that row is also true. Hence the argument form is valid.

Now consider another valid argument form called **modus tollens**. It has the following form:

If $p$ then $q$.
$\sim q$
$\therefore \sim p$

Here is an example of modus tollens:

If Zeus is human, then Zeus is mortal.
Zeus is not mortal.
∴ Zeus is not human.

An intuitive explanation for the validity of modus tollens uses proof by contradiction. It goes like this:

Suppose

(1) If Zeus is human, then Zeus is mortal; and
(2) Zeus is not mortal.

Must Zeus necessarily be nonhuman?

Yes!

Because, if Zeus were human, then by (1) he would be mortal.
But by (2) he is not mortal.
Hence, Zeus cannot be human.

*Modus tollens* is Latin meaning "method of denying" (the conclusion is a denial). The validity of modus tollens can be shown to follow from modus ponens together with the fact that a conditional statement is logically equivalent to its contrapositive. Or it can be established formally by using a truth table. (See exercise 13.)

Studies by cognitive psychologists have shown that although nearly 100% of college students have a solid, intuitive understanding of modus ponens, less than 60% are able to apply modus tollens correctly.* Yet in mathematical reasoning, modus tollens is used almost as often as modus ponens. Thus it is important to study the form of modus tollens carefully to learn to use it effectively.

### Example 2.3.2 Recognizing Modus Ponens and Modus Tollens

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

a. If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes.
∴ ______________________________.

b. If 870,232 is divisible by 6, then it is divisible by 3.
870,232 is not divisible by 3.
∴ ______________________________.

### Solution

a. At least two pigeons roost in the same hole. by modus ponens

b. 870,232 is not divisible by 6. by modus tollens ■

## Additional Valid Argument Forms: Rules of Inference

A **rule of inference** is a form of argument that is valid. Thus modus ponens and modus tollens are both rules of inference. The following are additional examples of rules of inference that are frequently used in deductive reasoning.

### Example 2.3.3 Generalization

The following argument forms are valid:

a. $p$
$\therefore p \vee q$

b. $q$
$\therefore p \vee q$

---

*[*Cognitive Psychology and Its Implications*, 3d ed. by John R. Anderson (New York: Freeman, 1990), pp. 292–297.

These argument forms are used for making generalizations. For instance, according to the first, if $p$ is true, then, more generally, "$p$ or $q$" is true for *any* other statement $q$. As an example, suppose you are given the job of counting the upperclassmen at your school. You ask what class Anton is in and are told he is a junior.

You reason as follows:

$$\begin{array}{l} \text{Anton is a junior.} \\ \therefore \text{ (more generally) Anton is a junior or Anton is a senior.} \end{array}$$

Knowing that upperclassman means junior or senior, you add Anton to your list. ■

## Example 2.3.4 Specialization

The following argument forms are valid:

$$\text{a.}\quad \begin{array}{l} p \wedge q \\ \therefore p \end{array} \qquad\qquad \text{b.}\quad \begin{array}{l} p \wedge q \\ \therefore q \end{array}$$

These argument forms are used for specializing. When classifying objects according to some property, you often know much more about them than whether they do or do not have that property. When this happens, you discard extraneous information as you concentrate on the particular property of interest.

For instance, suppose you are looking for a person who knows graph algorithms to work with you on a project. You discover that Ana knows both numerical analysis and graph algorithms. You reason as follows:

$$\begin{array}{l} \qquad\qquad \text{Ana knows numerical analysis and Ana knows graph algorithms.} \\ \therefore \text{ (in particular) Ana knows graph algorithms.} \end{array}$$

Accordingly, you invite her to work with you on your project. ■

Both generalization and specialization are used frequently in mathematics to tailor facts to fit into hypotheses of known theorems in order to draw further conclusions. Elimination, transitivity, and proof by division into cases are also widely used tools.

## Example 2.3.5 Elimination

The following argument forms are valid:

$$\text{a.}\quad \begin{array}{l} p \vee q \\ \sim q \\ \therefore p \end{array} \qquad\qquad \text{b.}\quad \begin{array}{l} p \vee q \\ \sim p \\ \therefore q \end{array}$$

These argument forms say that when you have only two possibilities and you can rule one out, the other must be the case. For instance, suppose you know that for a particular number $x$,

$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0.$$

If you also know that $x$ is not negative, then $x \neq -2$, so

$$x + 2 \neq 0.$$

By elimination, you can then conclude that

$$\therefore x - 3 = 0.$$

■

## Example 2.3.6 Transitivity

The following argument form is valid:

$$\begin{aligned} &p \to q \\ &q \to r \\ \therefore\ &p \to r \end{aligned}$$

Many arguments in mathematics contain chains of if-then statements. From the fact that one statement implies a second and the second implies a third, you can conclude that the first statement implies the third. In the example below suppose $n$ is a particular integer.

If $n$ is divisible by 18, then $n$ is divisible by 9.

If $n$ is divisible by 9, then the sum of the digits of $n$ is divisible by 9.

∴ If $n$ is divisible by 18, then the sum of the digits of $n$ is divisible by 9. ■

## Example 2.3.7 Proof by Division into Cases

The following argument form is valid:

$$\begin{aligned} &p \lor q \\ &p \to r \\ &q \to r \\ \therefore\ &r \end{aligned}$$

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true. For instance, suppose you know that $x$ is a particular nonzero real number that is not zero. The trichotomy property of the real numbers says that any real number is positive, negative, or zero. Thus (by elimination) you know that $x$ is positive or $x$ is negative. You can deduce that $x^2 > 0$ by arguing as follows:

$x$ is positive or $x$ is negative.

If $x$ is positive, then $x^2 > 0$.

If $x$ is negative, then $x^2 > 0$.

$\therefore x^2 > 0$. ■

The rules of valid inference are used constantly in problem solving. Here is an example from everyday life.

## Example 2.3.8 Application: A More Complex Deduction

You are about to leave for class in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading my class notes in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading my class notes in the living room or I was reading my class notes in the kitchen.

e. If I was reading my class notes in the living room then my glasses are on the coffee table.

Where are the glasses?

**Solution** Let $RK$ = I was reading my class notes in the kitchen.
$GK$ = My glasses are on the kitchen table.
$SB$ = I saw my glasses at breakfast.
$RL$ = I was reading my class notes in the living room.
$GC$ = My glasses are on the coffee table.

Here is a sequence of steps you might use to reach the answer, together with the rules of inference that allow you to draw the conclusion of each step:

1. $RK \to GK$ by (a)
   $GK \to SB$ by (b)
   $\therefore RK \to SB$ by transitivity

2. $RK \to SB$ by the conclusion of (1)
   $\sim SB$ by (c)
   $\therefore \sim RK$ by modus tollens

3. $RL \vee RK$ by (d)
   $\sim RK$ by the conclusion of (2)
   $\therefore RL$ by elimination

4. $RL \to GC$ by (e)
   $RL$ by the conclusion of (3)
   $\therefore GC$ by modus ponens

Thus the glasses are on the coffee table. ■

## Fallacies

A **fallacy** is an error in reasoning that results in an invalid argument. Three common fallacies are **using ambiguous premises**, and treating them as if they were unambiguous, **circular reasoning** (**assuming what is to be proved** without having derived it from the premises), and **jumping to a conclusion** (without adequate grounds). In this section we discuss two other fallacies, called *converse error* and *inverse error*, which give rise to arguments that superficially resemble those that are valid by modus ponens and modus tollens but are not, in fact, valid.

As in previous examples, you can show that an argument is invalid by constructing a truth table for the argument form and finding at least one critical row in which all the premises are true but the conclusion is false. Another way is to find an argument of the same form with true premises and a false conclusion.

For an argument to be valid, every argument of the same form whose premises are all true must have a true conclusion. It follows that for an argument to be invalid means that there is an argument of that form whose premises are all true and whose conclusion is false.

**Example 2.3.9**

### Converse Error

Show that the following argument is invalid:

If Zeke is a cheater, then Zeke sits in the back row.
Zeke sits in the back row.
∴ Zeke is a cheater.

**Solution** Many people recognize the invalidity of the above argument intuitively, reasoning something like this: The first premise gives information about Zeke *if* it is known he is a cheater. It doesn't give any information about him if it is not already known that he is a cheater. One can certainly imagine a person who is not a cheater but happens to sit in the back row. Then if that person's name is substituted for Zeke, the first premise is true by default and the second premise is also true but the conclusion is false.

The general form of the previous argument is as follows:

$$\begin{array}{l} p \to q \\ q \\ \therefore p \end{array}$$

In exercise 12(a) at the end of this section you are asked to use a truth table to show that this form of argument is invalid. ■

The fallacy underlying this invalid argument form is called the **converse error** because the conclusion of the argument would follow from the premises if the premise $p \to q$ were replaced by its converse. Such a replacement is not allowed, however, because a conditional statement is not logically equivalent to its converse. Converse error is also known as the *fallacy of affirming the consequent.*

A related common reasoning error is shown in the next example.

**Example 2.3.10**

### Inverse Error

Consider the following argument:

If these two vertices are adjacent, then they do not have the same color.
These two vertices are not adjacent.
∴ These two vertices have the same color.

Note that this argument has the following form:

$$\begin{array}{l} p \to q \\ \sim p \\ \therefore \sim q \end{array}$$

You are asked to give a truth table verification of the invalidity of this argument form in exercise 12(b) at the end of this section.

The fallacy underlying this invalid argument form is called the **inverse error** because the conclusion of the argument would follow from the premises if the premise $p \to q$ were replaced by its inverse. Such a replacement is not allowed, however, because a conditional statement is not logically equivalent to its inverse. Inverse error is also known as the *fallacy of denying the antecedent.* ■

**Caution!** In logic, the words *true* and *valid* have very different meanings. A valid argument may have a false conclusion, and an invalid argument may have a true conclusion.

Sometimes people lump together the ideas of validity and truth. If an argument seems valid, they accept the conclusion as true. And if an argument seems fishy (really a slang expression for invalid), they think the conclusion must be false. This is not correct!

### Example 2.3.11 A Valid Argument with a False Premise and a False Conclusion

The argument below is valid by modus ponens. But its major premise is false, and so is its conclusion.

If Canada is north of the United States, then temperatures in Canada never rise above freezing.
Canada is north of the United States.
∴ Temperatures in Canada never rise above freezing. ■

### Example 2.3.12 An Invalid Argument with True Premises and a True Conclusion

The argument below is invalid by the converse error, but it has a true conclusion.

If New York is a big city, then New York has tall buildings.
New York has tall buildings.
∴ New York is a big city. ■

**Definition**

An argument is called **sound** if, and only if, it is valid *and* all its premises are true. An argument that is not sound is called **unsound**.

The important thing to note is that validity is a property of argument *forms:* If an argument is valid, then so is every other argument that has the same form. Similarly, if an argument is invalid, then so is every other argument that has the same form. What characterizes a valid argument is that no argument whose form is valid can have all true premises and a false conclusion. For each valid argument, there are arguments of that form with all true premises and a true conclusion, with at least one false premise and a true conclusion, and with at least one false premise and a false conclusion. On the other hand, for each invalid argument, there are arguments of that form with every combination of truth values for the premises and conclusion, including all true premises and a false conclusion. The bottom line is that we can only be sure that the conclusion of an argument is true when we know that the argument is sound, that is, when we know both that the argument is valid and that it has all true premises.

## *Contradictions and Valid Arguments*

The concept of logical contradiction can be used to make inferences through a technique of reasoning called the *contradiction rule*. Suppose $p$ is some statement whose truth you wish to deduce.

**Contradiction Rule**

If you can show that the supposition that statement $p$ is false leads logically to a contradiction, then you can conclude that $p$ is true.

## Example 2.3.13 Contradiction Rule

Show that the following argument form is valid:

$$\begin{array}{l} \sim p \rightarrow \mathbf{c}, \text{ where } \mathbf{c} \text{ is a contradiction} \\ \therefore p \end{array}$$

**Solution** Construct a truth table for the premise and the conclusion of this argument.

| | | | premises | conclusion |
|---|---|---|---|---|
| $p$ | $\sim p$ | $\mathbf{c}$ | $\sim p \rightarrow \mathbf{c}$ | $p$ |
| T | F | F | T | T |
| F | T | F | F | |

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid. ■

The contradiction rule is the logical heart of the method of proof by contradiction. A slight variation also provides the basis for solving many logical puzzles by eliminating contradictory answers: *If an assumption leads to a contradiction, then that assumption must be false.*

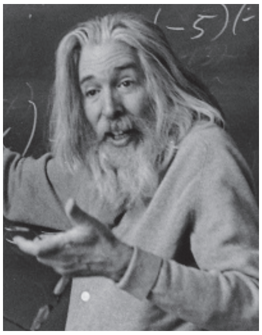
## Example 2.3.14 Knights and Knaves

The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie.* You visit the island and are approached by two natives who speak to you as follows:

*A* says: *B* is a knight.

*B* says: *A* and I are of opposite type.

What are *A* and *B*?



Eddie Hausner/The *New York Times*/Redux

*Raymond Smullyan (1919–2017)*

**Solution** *A* and *B* are both knaves. To see this, reason as follows:

| | |
|---|---|
| Suppose *A* is a knight. | |
| ∴ What *A* says is true. | by definition of knight |
| ∴ *B* is also a knight. | That's what *A* said. |
| ∴ What *B* says is true. | by definition of knight |
| ∴ *A* and *B* are of opposite types. | That's what *B* said. |
| ∴ We have arrived at the following contradiction: *A* and *B* are both knights and *A* and *B* are of opposite type. | |
| ∴ The supposition is false. | by the contradiction rule |
| ∴ *A* is not a knight. | negation of supposition |
| ∴ *A* is a knave. | by elimination: It's given that all inhabitants are knights or knaves, so since *A* is not a knight, *A* is a knave. |
| ∴ What *A* says is false. | |
| ∴ *B* is not a knight. | |
| ∴ *B* is also a knave. | by elimination |

---

*Raymond Smullyan has written a delightful series of whimsical yet profound books of logical puzzles starting with *What Is the Name of This Book?* (Englewood Cliffs, New Jersey: Prentice-Hall, 1978). Other good sources of logical puzzles are the many excellent books of Martin Gardner, such as *Aha! Insight* and *Aha! Gotcha* (New York: W. H. Freeman, 1978, 1982).

This reasoning shows that if the problem has a solution at all, then $A$ and $B$ must both be knaves. It is conceivable, however, that the problem has no solution. The problem statement could be inherently contradictory. If you look back at the solution, though, you can see that it does work out for both $A$ and $B$ to be knaves. ■

## Summary of Rules of Inference

Table 2.3.1 summarizes some of the most important rules of inference.

**TABLE 2.3.1** Valid Argument Forms

| | | | |
|---|---|---|---|
| **Modus Ponens** | $p \to q$<br>$p$<br>$\therefore q$ | **Elimination** | **a.** $p \lor q$<br>$\sim q$<br>$\therefore p$<br>**b.** $p \lor q$<br>$\sim p$<br>$\therefore q$ |
| **Modus Tollens** | $p \to q$<br>$\sim q$<br>$\therefore \sim p$ | **Transitivity** | $p \to q$<br>$q \to r$<br>$\therefore p \to r$ |
| **Generalization** | **a.** $p$<br>$\therefore p \lor q$<br>**b.** $q$<br>$\therefore p \lor q$ | **Proof by Division into Cases** | $p \lor q$<br>$p \to r$<br>$q \to r$<br>$\therefore r$ |
| **Specialization** | **a.** $p \land q$<br>$\therefore p$<br>**b.** $p \land q$<br>$\therefore q$ | | |
| **Conjunction** | $p$<br>$q$<br>$\therefore p \land q$ | **Contradiction Rule** | $\sim p \to \mathbf{c}$<br>$\therefore p$ |

## TEST YOURSELF

1. For an argument to be valid means that every argument of the same form whose premises _____ has a _____ conclusion.
2. For an argument to be invalid means that there is an argument of the same form whose premises _____ and whose conclusion _____.
3. For an argument to be sound means that it is _____ and its premises _____. In this case we can be sure that its conclusion _____.

## EXERCISE SET 2.3

Use modus ponens or modus tollens to fill in the blanks in the arguments of 1–5 so as to produce valid inferences.

**1.** If $\sqrt{2}$ is rational, then $\sqrt{2} = a/b$ for some integers $a$ and $b$.
It is not true that $\sqrt{2} = a/b$ for some integers $a$ and $b$.
$\therefore$ ______________________________.

**2.** If $1 - 0.99999\ldots$ is less than every positive real number, then it equals zero.
______________________________.
$\therefore$ The number $1 - 0.99999\ldots$ equals zero.

**3.** If logic is easy, then I am a monkey's uncle.
I am not a monkey's uncle.
$\therefore$ ______________________________.

**4.** If this graph can be colored with three colors, then it can colored with four colors.
This graph cannot be colored with four colors.
$\therefore$ ______________________________.

**5.** If they were unsure of the address, then they would have telephoned.
______________________________.
$\therefore$ They were sure of the address.

Use truth tables to determine whether the argument forms in 6–11 are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer. Your explanation should show that you understand what it means for a form of argument to be valid or invalid.

**6.** $p \to q$
$q \to p$
$\therefore p \vee q$

**7.** $p$
$p \to q$
$\sim q \vee r$
$\therefore r$

**8.** $p \vee q$
$p \to \sim q$
$p \to r$
$\therefore r$

**9.** $p \wedge q \to \sim r$
$p \vee \sim q$
$\sim q \to p$
$\therefore \sim r$

**10.** $p \vee q \to r$
$\therefore \sim r \to \sim p \wedge \sim q$
(This is the form of argument shown on pages 37 and 38.)

**11.** $p \to q \vee r$
$\sim q \vee \sim r$
$\therefore \sim p \vee \sim r$

**12.** Use truth tables to show that the following forms of argument are invalid.

**a.** $p \to q$
$q$
$\therefore p$
(converse error)

**b.** $p \to q$
$\sim p$
$\therefore \sim q$
(inverse error)

Use truth tables to show that the argument forms referred to in 13–21 are valid. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer. Your explanation should show that you understand what it means for a form of argument to be valid.

**13.** Modus tollens:
$p \to q$
$\sim q$
$\therefore \sim p$

**14.** Example 2.3.3(a)

**15.** Example 2.3.3(b)

**16.** Example 2.3.4(a)

**17.** Example 2.3.4(b)

**18.** Example 2.3.5(a)

**19.** Example 2.3.5(b)

**20.** Example 2.3.6

**21.** Example 2.3.7

Use symbols to write the logical form of each argument in 22 and 23, and then use a truth table to test the argument for validity. Indicate which columns represent the premises and which represent the conclusion, and include a few words of explanation showing that you understand the meaning of validity.

**22.** If Tom is not on team *A*, then Hua is on team *B*.
If Hua is not on team *B*, then Tom is on team *A*.
∴ Tom is not on team *A* or Hua is not on team *B*.

**23.** Oleg is a math major or Oleg is an economics major.
If Oleg is a math major, then Oleg is required to take Math 362.
∴ Oleg is an economics major or Oleg is not required to take Math 362.

Some of the arguments in 24–32 are valid, whereas others exhibit the converse or the inverse error. Use symbols to write the logical form of each argument. If the argument is valid, identify the rule of inference that guarantees its validity. Otherwise, state whether the converse or the inverse error is made.

**24.** If Jules solved this problem correctly, then Jules obtained the answer 2.
Jules obtained the answer 2.
∴ Jules solved this problem correctly.

**25.** This real number is rational or it is irrational.
This real number is not rational.
∴ This real number is irrational.

**26.** If I go to the movies, I won't finish my homework.
If I don't finish my homework, I won't do well on the exam tomorrow.
∴ If I go to the movies, I won't do well on the exam tomorrow.

**27.** If this number is larger than 2, then its square is larger than 4.
This number is not larger than 2.
∴ The square of this number is not larger than 4.

**28.** If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite.
The set of all irrational numbers is infinite.
∴ There are as many rational numbers as there are irrational numbers.

**29.** If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
Neither of these two numbers is divisible by 6.
∴ The product of these two numbers is not divisible by 6.

**30.** If this computer program is correct, then it produces the correct output when run with the test data my teacher gave me.
This computer program produces the correct output when run with the test data my teacher gave me.
∴ This computer program is correct.

**31.** Sandra knows Java and Sandra knows C++.
∴ Sandra knows C++.

**32.** If I get a Christmas bonus, I'll buy a stereo.
If I sell my motorcycle, I'll buy a stereo.
∴ If I get a Christmas bonus or I sell my motorcycle, then I'll buy a stereo.

**33.** Give an example (other than Example 2.3.11) of a valid argument with a false conclusion.

**34.** Give an example (other than Example 2.3.12) of an invalid argument with a true conclusion.

**35.** Explain in your own words what distinguishes a valid form of argument from an invalid one.

**36.** Given the following information about a computer program, find the mistake in the program.

- **a.** There is an undeclared variable or there is a syntax error in the first five lines.
- **b.** If there is a syntax error in the first five lines, then there is a missing semicolon or a variable name is misspelled.
- **c.** There is not a missing semicolon.
- **d.** There is not a misspelled variable name.

**37.** In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humor and love of logical puzzles. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (a–e below) and challenged the reader to use them to figure out the location of the treasure.

- **a.** If this house is next to a lake, then the treasure is not in the kitchen.
- **b.** If the tree in the front yard is an elm, then the treasure is in the kitchen.
- **c.** This house is next to a lake.
- **d.** The tree in the front yard is an elm or the treasure is buried under the flagpole.
- **e.** If the tree in the back yard is an oak, then the treasure is in the garage.

Where is the treasure hidden?

**38.** You are visiting the island described in Example 2.3.14 and have the following encounters with natives.

- **a.** Two natives $A$ and $B$ address you as follows:
  $A$ says: Both of us are knights.
  $B$ says: $A$ is a knave.
  What are $A$ and $B$?
- **b.** Another two natives $C$ and $D$ approach you but only $C$ speaks.
  $C$ says: Both of us are knaves.
  What are $C$ and $D$?
- **c.** You then encounter natives $E$ and $F$.
  $E$ says: $F$ is a knave.
  $F$ says: $E$ is a knave.
  How many knaves are there?
- *H* **d.** Finally, you meet a group of six natives, $U$, $V$, $W$, $X$, $Y$, and $Z$, who speak to you as follows:
  $U$ says: None of us is a knight.
  $V$ says: At least three of us are knights.
  $W$ says: At most three of us are knights.
  $X$ says: Exactly five of us are knights.
  $Y$ says: Exactly two of us are knights.
  $Z$ says: Exactly one of us is a knight.
  Which are knights and which are knaves?

**39.** The famous detective Percule Hoirot was called in to solve a baffling murder mystery. He determined the following facts:

- **a.** Lord Hazelton, the murdered man, was killed by a blow on the head with a brass candlestick.
- **b.** Either Lady Hazelton or a maid, Sara, was in the dining room at the time of the murder.
- **c.** If the cook was in the kitchen at the time of the murder, then the butler killed Lord Hazelton with a fatal dose of strychnine.
- **d.** If Lady Hazelton was in the dining room at the time of the murder, then the chauffeur killed Lord Hazelton.
- **e.** If the cook was not in the kitchen at the time of the murder, then Sara was not in the dining room when the murder was committed.
- **f.** If Sara was in the dining room at the time the murder was committed, then the wine steward killed Lord Hazelton.

Is it possible for the detective to deduce the identity of the murderer from these facts? If so, who did murder Lord Hazelton? (Assume there was only one cause of death.)

**40.** Sharky, a leader of the underworld, was killed by one of his own band of four henchmen. Detective Sharp interviewed the men and determined that all were lying except for one. He deduced who killed Sharky on the basis of the following statements:

- **a.** Socko: Lefty killed Sharky.
- **b.** Fats: Muscles didn't kill Sharky.

- **c.** Lefty: Muscles was shooting craps with Socko when Sharky was knocked off.
- **d.** Muscles: Lefty didn't kill Sharky.

Who did kill Sharky?

In 41–44 a set of premises and a conclusion are given. Use the valid argument forms listed in Table 2.3.1 to deduce the conclusion from the premises, giving a reason for each step as in Example 2.3.8. Assume all variables are statement variables.

**41.** **a.** $\sim p \lor q \to r$
**b.** $s \lor \sim q$
**c.** $\sim t$
**d.** $p \to t$
**e.** $\sim p \land r \to \sim s$
**f.** $\therefore \sim q$

**42.** **a.** $p \lor q$
**b.** $q \to r$
**c.** $p \land s \to t$
**d.** $\sim r$
**e.** $\sim q \to u \land s$
**f.** $\therefore t$

**43.** **a.** $\sim p \to r \land \sim s$
**b.** $t \to s$
**c.** $u \to \sim p$
**d.** $\sim w$
**e.** $u \lor w$
**f.** $\therefore \sim t$

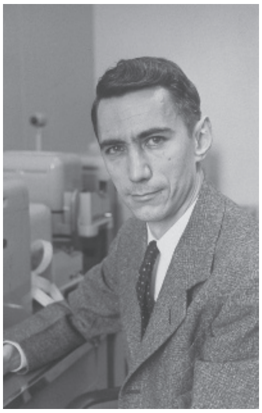
**44.** **a.** $p \to q$
**b.** $r \lor s$
**c.** $\sim s \to \sim t$
**d.** $\sim q \lor s$
**e.** $\sim s$
**f.** $\sim p \land r \to u$
**g.** $w \lor t$
**h.** $\therefore u \land w$

## ANSWERS FOR TEST YOURSELF

**1.** are all true; true **2.** are all true; is false **3.** valid; are all true; is true

# 2.4 Application: Digital Logic Circuits

*Only connect!* —E. M. Forster, *Howards End*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*Claude Shannon (1916–2001)*

In the late 1930s, a young M.I.T. graduate student named Claude Shannon noticed an analogy between the operations of switching devices, such as telephone switching circuits, and the operations of logical connectives. He used this analogy with striking success to solve problems of circuit design and wrote up his results in his master's thesis, which was published in 1938.

The drawing in Figure 2.4.1(a) shows the appearance of the two positions of a simple switch. When the switch is closed, current can flow from one terminal to the other; when it is open, current cannot flow. Imagine that such a switch is part of the circuit shown in Figure 2.4.1(b). The light bulb turns on if, and only if, current flows through it. And this happens if, and only if, the switch is closed.

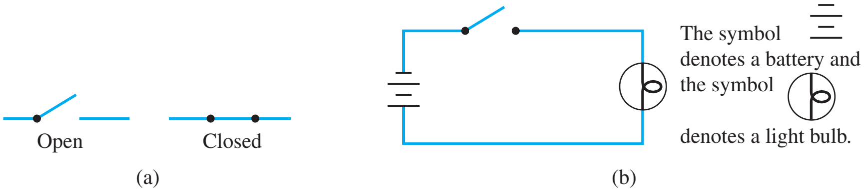


**FIGURE 2.4.1**

Now consider the more complicated circuits of Figures 2.4.2(a) and 2.4.2(b).

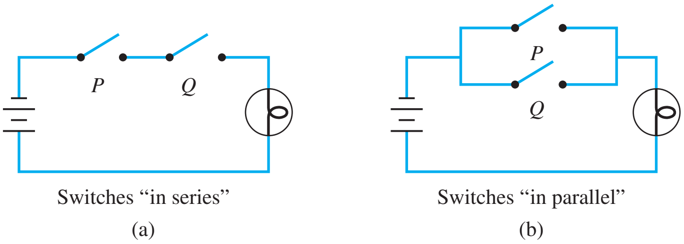


**FIGURE 2.4.2**

In the circuit of Figure 2.4.2(a) current flows and the light bulb turns on if, and only if, *both* switches $P$ and $Q$ are closed. The switches in this circuit are said to be **in series**. In the circuit of Figure 2.4.2(b) current flows and the light bulb turns on if, and only if, *at least one* of the switches $P$ or $Q$ is closed. The switches in this circuit are said to be **in parallel**. All possible behaviors of these circuits are described by Table 2.4.1.

**TABLE 2.4.1**

(a) Switches in Series

| Switches | | Light Bulb |
|---|---|---|
| ***P*** | ***Q*** | **State** |
| closed | closed | on |
| closed | open | off |
| open | closed | off |
| open | open | off |

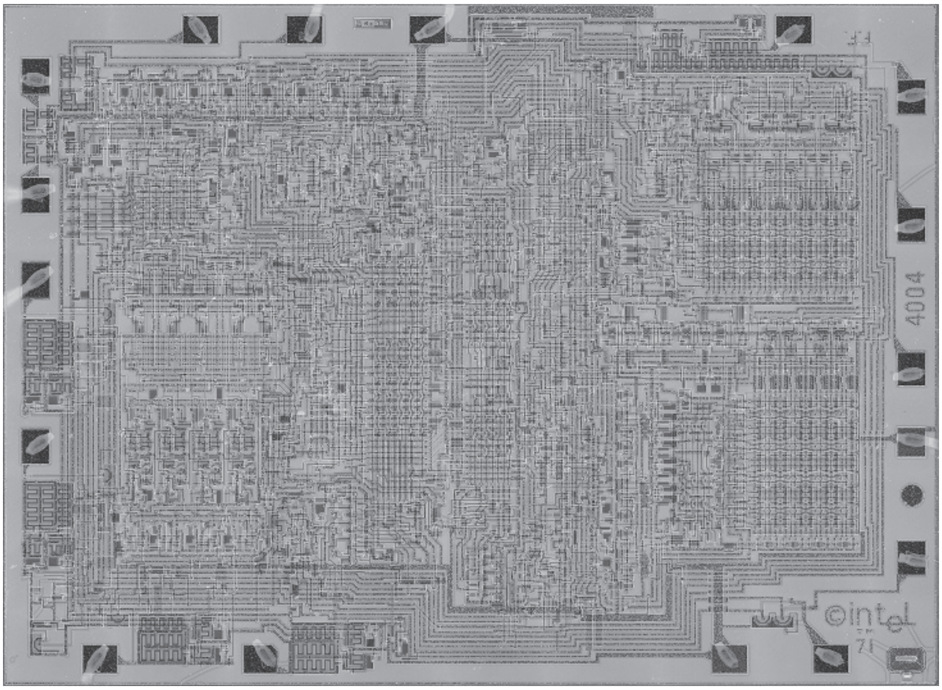
(b) Switches in Parallel

| Switches | | Light Bulb |
|---|---|---|
| ***P*** | ***Q*** | **State** |
| closed | closed | on |
| closed | open | on |
| open | closed | on |
| open | open | off |

Observe that if the words *closed* and *on* are replaced by T and *open* and *off* are replaced by F, Table 2.4.1(a) becomes the truth table for *and* and Table 2.4.1(b) becomes the truth table for *or*. Consequently, the switching circuit of Figure 2.4.2(a) is said to correspond to the logical expression $P \wedge Q$, and that of Figure 2.4.2(b) is said to correspond to $P \vee Q$.

More complicated circuits correspond to more complicated logical expressions. This correspondence has been used extensively in the design and study of circuits.

In the 1940s and 1950s, switches were replaced by electronic devices, with the physical states of closed and open corresponding to electronic states such as high and low voltages.



Tim McNerney

*The Intel 4004, introduced in 1971, is generally considered to be the first commercially viable microprocessor or central processing unit (CPU) contained on a chip about the size of a fingernail. It consisted of 2,300 transistors and could execute 70,000 instructions per second, essentially the same computing power as the first electronic computer, the ENIAC, built in 1946, which filled an entire room. Modern microprocessors consist of several CPUs on one chip, contain close to a billion transistors and many hundreds of millions of logic circuits, and can compute hundreds of millions of instructions per second.*

Alfred Eisenstaedt/Time Life Pictures/Getty Images

*John W. Tukey (1915–2000)*

The new electronic technology led to the development of modern digital systems such as electronic computers, electronic telephone switching systems, traffic light controls, electronic calculators, and the control mechanisms used in hundreds of other types of electronic equipment. The basic electronic components of a digital system are called *digital logic circuits*. The word *logic* indicates the important role of logic in the design of such circuits, and the word *digital* indicates that the circuits process discrete, or separate, signals as opposed to continuous ones.

Electrical engineers continue to use the language of logic when they refer to values of signals produced by an electronic switch as being "true" or "false." But they generally use the symbols 1 and 0 rather than T and F to denote these values. The symbols 0 and 1 are called **bits**, short for *b*inary dig*its*. This terminology was introduced in 1946 by the statistician John Tukey.

## Black Boxes and Gates

Combinations of signal bits (1's and 0's) can be transformed into other combinations of signal bits (1's and 0's) by means of various circuits. Because a variety of different technologies are used in circuit construction, computer engineers and digital system designers find it useful to think of certain basic circuits as black boxes. The inside of a black box contains the detailed implementation of the circuit and is often ignored while attention is focused on the relation between the **input** and the **output** signals.

The operation of a black box is completely specified by constructing an **input/output table** that lists all its possible input signals together with their corresponding output signals. For example, the black box pictured above has three input signals. Since each of these signals can take the value 1 or 0, there are eight possible combinations of input signals. One possible correspondence of input to output signals is as follows:

An Input/Output Table

| **Input** | | | **Output** |
|---|---|---|---|
| ***P*** | ***Q*** | ***R*** | ***S*** |
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 0 |

The third row, for instance, indicates that for inputs $P = 1$, $Q = 0$, and $R = 1$, the output $S$ is 0.

An efficient method for designing more complicated circuits is to build them by connecting less complicated black box circuits. Three such circuits are known as NOT-, AND-, and OR-gates.

A **NOT-gate** (or **inverter**) is a circuit with one input signal and one output signal. If the input signal is 1, the output signal is 0. Conversely, if the input signal is 0, then the output signal is 1. An **AND-gate** is a circuit with two input signals and one output signal. If both input signals are 1, then the output signal is 1. Otherwise, the output signal is 0. An **OR-gate** also has two input signals and one output signal. If both input signals are 0, then the output signal is 0. Otherwise, the output signal is 1.

The actions of NOT-, AND-, and OR-gates are summarized in Figure 2.4.3, where $P$ and $Q$ represent input signals and $R$ represents the output signal. It should be clear from Figure 2.4.3 that the actions of the NOT-, AND-, and OR-gates on signals correspond exactly to those of the logical connectives $\sim$, $\wedge$, and $\vee$ on statements, if the symbol 1 is identified with T and the symbol 0 is identified with F.

Gates can be combined into circuits in a variety of ways. If the rules shown at the bottom of the page are obeyed, the result is a **combinational circuit**, one whose output at any time is determined entirely by its input at that time without regard to previous inputs.

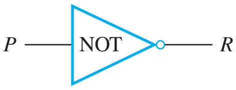
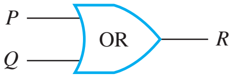
<table>
<tr><th>Type of Gate</th><th>Symbolic Representation</th><th>Action</th></tr>
<tr><td>NOT</td><td>
</td><td><table>
<tr><th>Input</th><th>Output</th></tr>
<tr><th><em>P</em></th><th><em>R</em></th></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>0</td><td>1</td></tr>
</table></td></tr>
<tr><td>AND</td><td>
</td><td><table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th><em>P</em></th><th><em>Q</em></th><th><em>R</em></th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table></td></tr>
<tr><td>OR</td><td>
</td><td><table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th><em>P</em></th><th><em>Q</em></th><th><em>R</em></th></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
</table></td></tr>
</table>

FIGURE 2.4.3

### Rules for a Combinational Circuit

Never combine two input wires. 2.4.1

A single input wire can be split partway and used as input for two separate gates. 2.4.2

An output wire can be used as input. 2.4.3

No output of a gate can eventually feed back into that gate. 2.4.4

Rule (2.4.4) is violated in more complex circuits, called **sequential circuits**, whose output at any given time depends both on the input at that time and also on previous inputs. These circuits are discussed in Section 12.2.

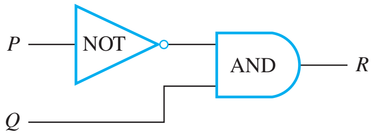
## *The Input/Output Table for a Circuit*

If you are given a set of input signals for a circuit, you can find its output by tracing through the circuit gate by gate.

### Example 2.4.1 Determining Output for a Given Input

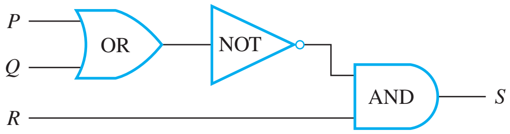
Indicate the output of the circuits shown below for the given input signals.

a.



Input signals: $P = 0$ and $Q = 1$

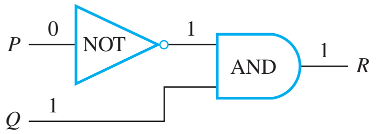
b.



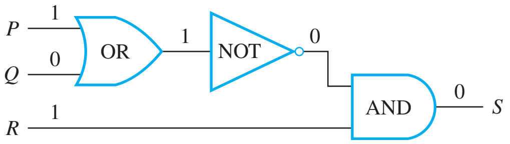
Input signals: $P = 1$, $Q = 0$, $R = 1$

### Solution

a. Move from left to right through the diagram, tracing the action of each gate on the input signals. The NOT-gate changes $P = 0$ to a 1, so both inputs to the AND-gate are 1; hence the output $R$ is 1. This is illustrated by annotating the diagram as shown below.



b. The output of the OR-gate is 1 since one of the input signals, $P$, is 1. The NOT-gate changes this 1 into a 0, so the two inputs to the AND-gate are 0 and $R = 1$. Hence the output $S$ is 0. The trace is shown below.

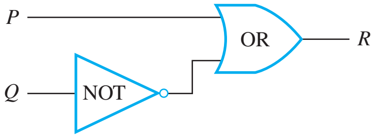


■

To construct the entire input/output table for a circuit, trace through the circuit to find the corresponding output signals for each possible combination of input signals.

### Example 2.4.2 Constructing the Input/Output Table for a Circuit

Construct the input/output table for the following circuit.



Illustrated London News Ltd./ Pantheon/Superstock

*George Boole*
*(1815–1864)*

**Solution** List the four possible combinations of input signals, and find the output for each by tracing through the circuit.

| Input | | Output |
|---|---|---|
| ***P*** | ***Q*** | ***R*** |
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

■

## *The Boolean Expression Corresponding to a Circuit*

In logic, variables such as $p$, $q$, and $r$ represent statements, and a statement can have one of only two truth values: T (true) or F (false). A statement form is an expression, such as $p \wedge (\sim q \vee r)$, composed of statement variables and logical connectives.

As noted earlier, one of the founders of symbolic logic was the English mathematician George Boole. In his honor, any variable, such as a statement variable or an input signal, that can take one of only two values is called a **Boolean variable**. An expression composed of Boolean variables and the connectives $\sim$, $\wedge$, and $\vee$ is called a **Boolean expression**.

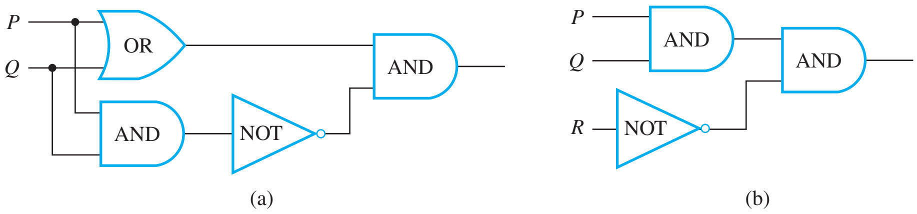
Given a circuit consisting of combined NOT-, AND-, and OR-gates, a corresponding Boolean expression can be obtained by tracing the actions of the gates on the input variables.

**Note** Strictly speaking, only meaningful expressions such as $(\sim p \wedge q) \vee (p \wedge r)$ and $\sim(\sim(p \wedge q) \vee r)$ are allowed as Boolean, not meaningless ones like $p \sim q((rs \vee \ \wedge q\sim$. We use recursion to give a careful definition of Boolean expressions in Section 5.9.

**Example 2.4.3**

### Finding a Boolean Expression for a Circuit

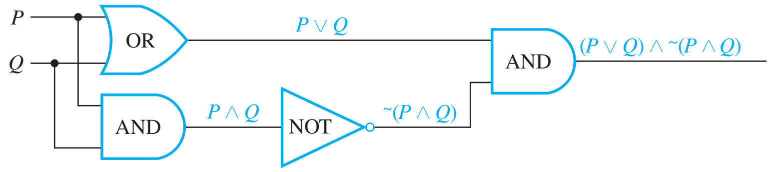
Find the Boolean expressions that correspond to the circuits shown below. A black dot indicates a soldering of two wires; wires that cross without a dot are assumed not to touch.



(a) (b)

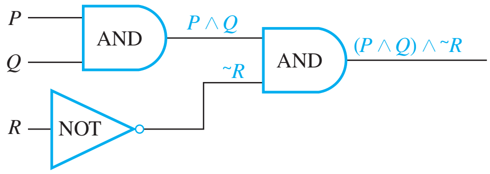
**Solution**

a. Trace through the circuit from left to right, indicating the output of each gate symbolically, as shown below.



The final expression obtained, $(P \vee Q) \wedge \sim(P \wedge Q)$, is the expression for exclusive or: $P$ or $Q$ but not both.

b. The Boolean expression corresponding to the circuit is $(P \wedge Q) \wedge \sim R$, as shown on the next page.



■

Observe that the output of the circuit shown in Example 2.4.3(b) is 1 for exactly one combination of inputs ($P = 1$, $Q = 1$, and $R = 0$) and is 0 for all other combinations of inputs. For this reason, the circuit can be said to "recognize" one particular combination of inputs. The output column of the input/output table has a 1 in exactly one row and 0's in all other rows.

**Definition**

A **recognizer** is a circuit that outputs a 1 for exactly one particular combination of input signals and outputs 0's for all other combinations.

Input/Output Table for a Recognizer

| $P$ | $Q$ | $R$ | $(P \wedge Q) \wedge \sim R$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 |

## The Circuit Corresponding to a Boolean Expression

The preceding examples showed how to find a Boolean expression corresponding to a circuit. The following example shows how to construct a circuit corresponding to a Boolean expression. The strategy is to work from the outermost part of the Boolean expression to the innermost part, adding logic gates that correspond to the operations in the expression as you move from right to left in the circuit diagram.

**Example 2.4.4**

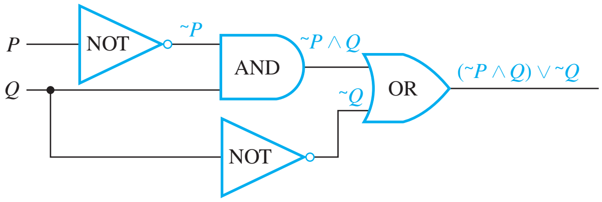
### Constructing Circuits for Boolean Expressions

Construct circuits for the following Boolean expressions.

a. $(\sim P \wedge Q) \vee \sim Q$ b. $((P \wedge Q) \wedge (R \wedge S)) \wedge T$

**Solution**

a. Write the input variables in a column on the left side of the diagram. Since the last operation executed when evaluating $(\sim P \wedge Q) \vee \sim Q$ is $\vee$, put an OR-gate at the extreme right of the diagram. One input to this gate is $\sim P \wedge Q$, so draw an AND-gate to the left of the OR-gate and show its output coming into the OR-gate. Since one input to the AND-gate is $\sim P$, draw a line from $P$ to a NOT-gate and from there to the AND-gate. Since the other input to the AND-gate is $Q$, draw a line from $Q$ directly to the AND-gate. The other input to the OR-gate is $\sim Q$, so draw a line from $Q$ to a NOT-gate and from the NOT-gate to the OR-gate. The circuit you obtain is shown below.



b. To start constructing this circuit, put one AND-gate at the extreme right to correspond to the $\wedge$, which is the final operation between $((P \wedge Q) \wedge (R \wedge S))$ and $T$. To the left of that gate put the AND-gate corresponding to the $\wedge$ between $P \wedge Q$ and $R \wedge S$. To the left of that gate put the two AND-gates corresponding to the $\wedge$'s between $P$ and $Q$ and between $R$ and $S$. The circuit is shown in Figure 2.4.4.

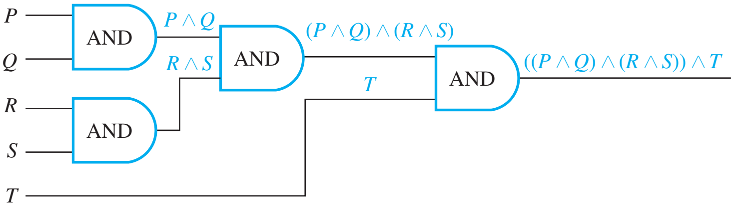


FIGURE 2.4.4

■

It follows from Theorem 2.1.1 that all the ways of adding parentheses to $P \wedge Q \wedge R \wedge S \wedge T$ give logically equivalent results. Thus, for example,

$$((P \wedge Q) \wedge (R \wedge S)) \wedge T \equiv (P \wedge (Q \wedge R)) \wedge (S \wedge T),$$

and hence the circuit in Figure 2.4.5, which corresponds to $(P \wedge (Q \wedge R)) \wedge (S \wedge T)$, has the same input/output table as the circuit in Figure 2.4.4, which corresponds to $((P \wedge Q) \wedge (R \wedge S)) \wedge T$.

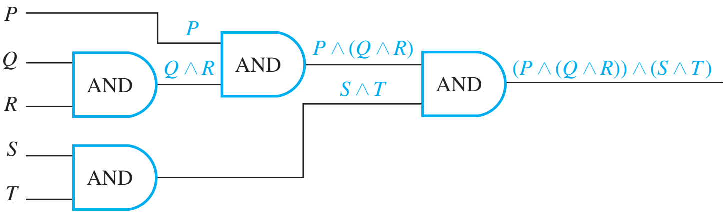


FIGURE 2.4.5

It follows that the circuits in Figures 2.4.4 and 2.4.5 are both implementations of the expression $P \wedge Q \wedge R \wedge S \wedge T$. Such a circuit is called a **multiple-input AND-gate** and is represented by the diagram shown in Figure 2.4.6. **Multiple-input OR-gates** are constructed similarly.

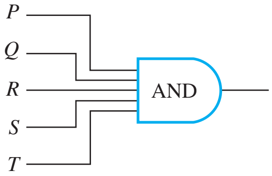


FIGURE 2.4.6

## *Finding a Circuit That Corresponds to a Given Input/Output Table*

To this point, we have discussed how to construct the input/output table for a circuit, how to find the Boolean expression corresponding to a given circuit, and how to construct the circuit corresponding to a given Boolean expression. Now we address the question of how to design a circuit (or find a Boolean expression) corresponding to a given input/output table. The way to do this is to put several recognizers together in parallel.

**Example 2.4.5**

### Designing a Circuit for a Given Input/Output Table

Design a circuit for the following input/output table:

| Input | | | Output |
|---|---|---|---|
| $P$ | $Q$ | $R$ | $S$ |
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 |

**Solution** First construct a Boolean expression with this table as its truth table. To do this, identify each row for which the output is 1—in this case, the rows 1, 3, and 4. For each such row, construct an *and* expression that produces a 1 (or true) for the exact combination of input values for that row and a 0 (or false) for all other combinations of input values.

For example, the expression for row 1 is $\boldsymbol{P \wedge Q \wedge R}$ because $P \wedge Q \wedge R$ is 1 if $P = 1$ and $Q = 1$ and $R = 1$, and it is 0 for all other values of $P$, $Q$, and $R$. The expression for row 3 is $\boldsymbol{P \wedge {\sim}Q \wedge R}$ because $P \wedge {\sim}Q \wedge R$ is 1 if $P = 1$ and $Q = 0$ and $R = 1$, and it is 0 for all other values of $P$, $Q$, and $R$. Similarly, the expression for row 4 is $\boldsymbol{P \wedge {\sim}Q \wedge {\sim}R}$.

Now any Boolean expression with the given table as its truth table has the value 1 in case $P \wedge Q \wedge R = 1$, or in case $P \wedge {\sim}Q \wedge R = 1$, or in case $P \wedge {\sim}Q \wedge {\sim}R = 1$, and in no other cases. It follows that a Boolean expression with the given truth table is

$$(P \wedge Q \wedge R) \vee (P \wedge {\sim}Q \wedge R) \vee (P \wedge {\sim}Q \wedge {\sim}R). \qquad 2.4.5$$

The circuit corresponding to this expression has the diagram shown in Figure 2.4.7. Observe that expression (2.4.5) is a disjunction of terms that are themselves conjunctions in which one of $P$ or $\sim P$, one of $Q$ or $\sim Q$, and one of $R$ or $\sim R$ all appear. Such expressions are said to be in **disjunctive normal form** or **sum-of-products form**.

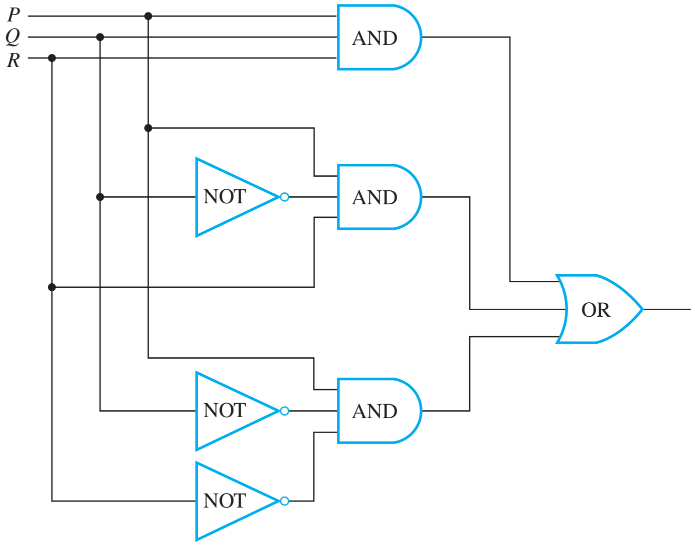
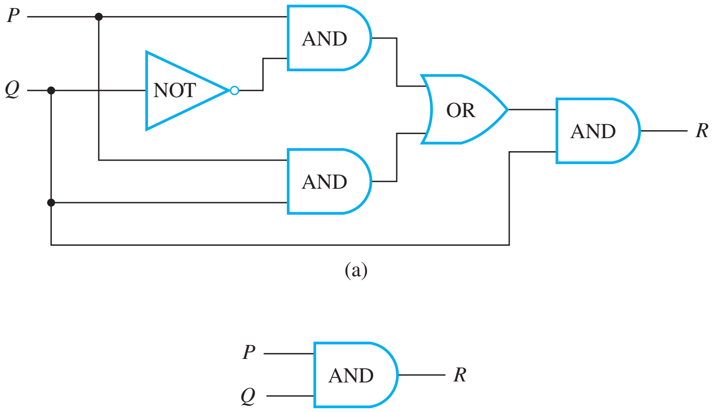


FIGURE 2.4.7 ■

## Simplifying Combinational Circuits

Consider the two combinational circuits shown in Figure 2.4.8.



(a)

(b)

FIGURE 2.4.8

If you trace through circuit (a), you will find that its input/output table is

| Input | | Output |
|---|---|---|
| ***P*** | ***Q*** | ***R*** |
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

which is the same as the input/output table for circuit (b). Thus these two circuits do the same job in the sense that they transform the same combinations of input signals into the same output signals. Yet circuit (b) is simpler than circuit (a) in that it contains many fewer logic gates. Thus, as part of an integrated circuit, it would take less space and require less power.

**Definition**

Two digital logic circuits are **equivalent** if, and only if, their input/output tables are identical.

Since logically equivalent statement forms have identical truth tables, you can determine that two circuits are equivalent by finding the Boolean expressions corresponding to the circuits and showing that these expressions, regarded as statement forms, are logically equivalent. Example 2.4.6 shows how this procedure works for circuits (a) and (b) in Figure 2.4.8.

### Example 2.4.6 Showing That Two Circuits Are Equivalent

Find the Boolean expressions for each circuit in Figure 2.4.8. Use Theorem 2.1.1 to show that these expressions are logically equivalent when regarded as statement forms.

**Solution** The Boolean expressions that correspond to circuits (a) and (b) are $((P \wedge \sim Q) \vee (P \wedge Q)) \wedge Q$ and $P \wedge Q$, respectively. By Theorem 2.1.1,

$$
\begin{aligned}
&((P \wedge \sim Q) \vee (P \wedge Q)) \wedge Q \\
&\quad \equiv (P \wedge (\sim Q \vee Q)) \wedge Q && \text{by the distributive law} \\
&\quad \equiv (P \wedge (Q \vee \sim Q)) \wedge Q && \text{by the commutative law for } \vee \\
&\quad \equiv (P \wedge \mathbf{t}) \wedge Q && \text{by the negation law} \\
&\quad \equiv P \wedge Q && \text{by the identity law.}
\end{aligned}
$$

It follows that the truth tables for $((P \wedge \sim Q) \vee (P \wedge Q)) \wedge Q$ and $P \wedge Q$ are the same. Hence the input/output tables for the circuits corresponding to these expressions are also the same, and so the circuits are equivalent. ■

In general, you can simplify a combinational circuit by finding the corresponding Boolean expression, using the properties listed in Theorem 2.1.1 to find a Boolean expression that is shorter and logically equivalent to it (when both are regarded as statement forms), and constructing the circuit corresponding to this shorter Boolean expression.

## NAND and NOR Gates

HUD 305.25. Harvard University Archives

*H. M. Sheffer (1882–1964)*

Another way to simplify a circuit is to find an equivalent circuit that uses the least number of different kinds of logic gates. Two gates not previously introduced are particularly useful for this: NAND-gates and NOR-gates. A NAND-gate is a single gate that acts like an AND-gate followed by a NOT-gate. A NOR-gate acts like an OR-gate followed by a NOT-gate. Thus the output signal of a NAND-gate is 0 when, and only when, both input signals are 1, and the output signal for a NOR-gate is 1 when, and only when, both input signals are 0. The logical symbols corresponding to these gates are $|$ (for NAND) and $\downarrow$ (for NOR), where $|$ is called a **Sheffer stroke** (after H. M. Sheffer, 1882–1964) and $\downarrow$ is called a **Peirce arrow** (after C. S. Peirce, 1839–1914; see page 110). Thus

$$P \,|\, Q \equiv \sim(P \wedge Q) \quad \text{and} \quad P \downarrow Q \equiv \sim(P \vee Q).$$

The table below summarizes the actions of NAND and NOR gates.

<table>
<tr><th>Type of Gate</th><th>Symbolic Representation</th><th>Action</th></tr>
<tr><td>NAND</td><td> </td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R = P | Q</th></tr>
<tr><td>1</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
</table>
</td></tr>
<tr><td>NOR</td><td> </td><td>
<table>
<tr><th colspan="2">Input</th><th>Output</th></tr>
<tr><th>P</th><th>Q</th><th>R = P ↓ Q</th></tr>
<tr><td>1</td><td>1</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
</table>
</td></tr>
</table>

It can be shown that any Boolean expression is equivalent to one written entirely with Sheffer strokes or entirely with Peirce arrows. Thus any digital logic circuit is equivalent to one that uses only NAND-gates or only NOR-gates. Example 2.4.7 develops part of the derivation of this result; the rest is left for the exercises.

### Example 2.4.7 Rewriting Expressions Using the Sheffer Stroke

Use Theorem 2.1.1 and the definition of Sheffer stroke to show that

a. $\sim P \equiv P|P$ and b. $P \vee Q \equiv (P|P)|(Q|Q)$.

### Solution

a. $\sim P \equiv \sim(P \wedge P)$ by the idempotent law for $\wedge$

$\equiv P|P$ by definition of $|$.

b. $P \vee Q \equiv \sim(\sim(P \vee Q))$ by the double negative law

$\equiv \sim(\sim P \wedge \sim Q)$ by De Morgan's laws

$\equiv \sim((P|P) \wedge (Q|Q))$ by part (a)

$\equiv (P|P)|(Q|Q)$ by definition of $|$. ■

## TEST YOURSELF

1. The input/output table for a digital logic circuit is a table that shows ______.
2. The Boolean expression that corresponds to a digital logic circuit is ______.
3. A recognizer is a digital logic circuit that ______.
4. Two digital logic circuits are equivalent if, and only if, ______.

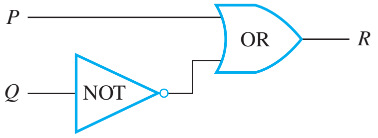
**5.** A NAND-gate is constructed by placing a ________ gate immediately following an ________ gate.

**6.** A NOR-gate is constructed by placing a ________ gate immediately following an ________ gate.

## EXERCISE SET 2.4

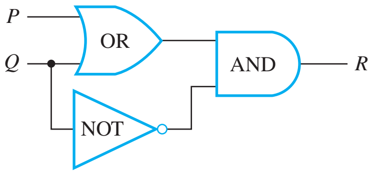
**Give the output signals for the circuits in 1–4 if the input signals are as indicated.**

**1.**



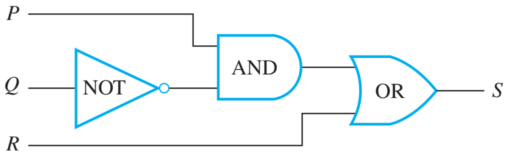
input signals: $P = 1$ and $Q = 1$

**2.**



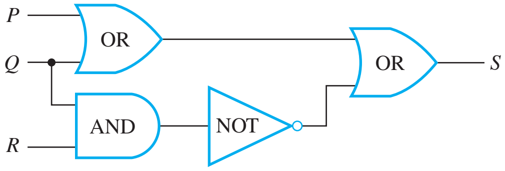
input signals: $P = 1$ and $Q = 0$

**3.**



input signals: $P = 1$, $Q = 0$, $R = 0$

**4.**



input signals: $P = 0$, $Q = 0$, $R = 0$

**In 5–8, write an input/output table for the circuit in the referenced exercise.**

**5.** Exercise 1
**6.** Exercise 2
**7.** Exercise 3
**8.** Exercise 4

**In 9–12, find the Boolean expression that corresponds to the circuit in the referenced exercise.**

**9.** Exercise 1
**10.** Exercise 2
**11.** Exercise 3
**12.** Exercise 4

**Construct circuits for the Boolean expressions in 13–17.**

**13.** $\sim P \vee Q$
**14.** $\sim(P \vee Q)$
**15.** $P \vee (\sim P \wedge \sim Q)$
**16.** $(P \wedge Q) \vee \sim R$
**17.** $(P \wedge \sim Q) \vee (\sim P \wedge R)$

**For each of the tables in 18–21, construct (a) a Boolean expression having the given table as its truth table and (b) a circuit having the given table as its input/output table.**

**18.**

| *P* | *Q* | *R* | *S* |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 |

**19.**

| *P* | *Q* | *R* | *S* |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 |

**20.**

| *P* | *Q* | *R* | *S* |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 |

**21.**

| $P$ | $Q$ | $R$ | $S$ |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 |

**22.** Design a circuit to take input signals $P$, $Q$, and $R$ and output a 1 if, and only if, $P$ and $Q$ have the same value and $Q$ and $R$ have opposite values.

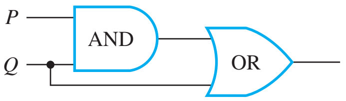
**23.** Design a circuit to take input signals $P$, $Q$, and $R$ and output a 1 if, and only if, all three of $P$, $Q$, and $R$ have the same value.

**24.** The lights in a classroom are controlled by two switches: one at the back of the room and one at the front. Moving either switch to the opposite position turns the lights off if they are on and on if they are off. Assume the lights have been installed so that when both switches are in the down position, the lights are off. Design a circuit to control the switches.

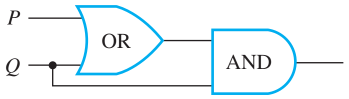
**25.** An alarm system has three different control panels in three different locations. To enable the system, switches in at least two of the panels must be in the on position. If fewer than two are in the on position, the system is disabled. Design a circuit to control the switches.

**Use the properties listed in Theorem 2.1.1 to show that each pair of circuits in 26–29 have the same input/output table. (Find the Boolean expressions for the circuits and show that they are logically equivalent when regarded as statement forms.)**

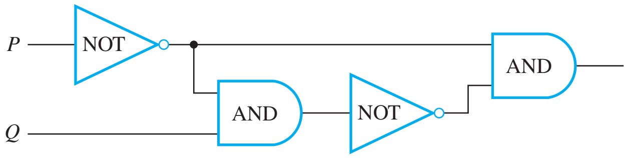
**26. a.**



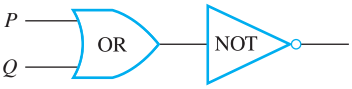
**b.**



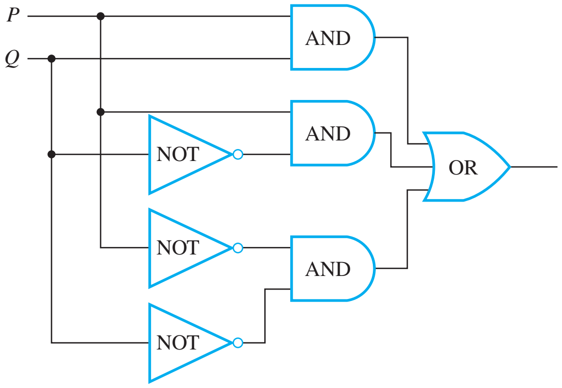
**27. a.**



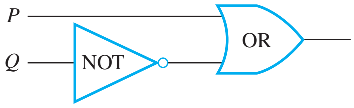
**b.**



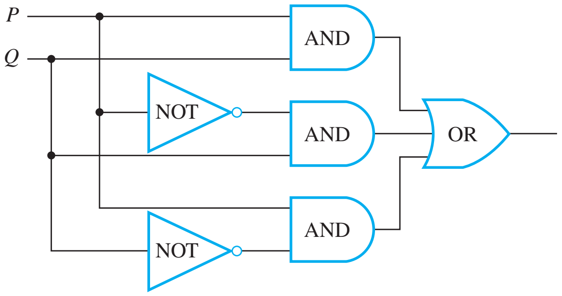
**28. a.**



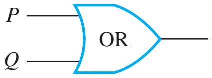
**b.**



**29. a.**



**b.**



**For the circuits corresponding to the Boolean expressions in each of 30 and 31 there is an equivalent circuit with at most two logic gates. Find such a circuit.**

**30.** $(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

**31.** $(\sim P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (P \wedge \sim Q)$

**32.** The Boolean expression for the circuit in Example 2.4.5 is

$$(P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R)$$

(a disjunctive normal form). Find a circuit with at most three logic gates that is equivalent to this circuit.

**33. a.** Show that for the Sheffer stroke $|$,

$$P \wedge Q \equiv (P|Q)|(P|Q).$$

**b.** Use the results of Example 2.4.7 and part (a) above to write $P \wedge (\sim Q \vee R)$ using only Sheffer strokes.

**34.** Show that the following logical equivalences hold for the Peirce arrow $\downarrow$, where $P \downarrow Q \equiv \sim(P \vee Q)$.

- **a.** $\sim P \equiv P \downarrow P$
- **b.** $P \vee Q \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$
- **c.** $P \wedge Q \equiv (P \downarrow P) \downarrow (Q \downarrow Q)$
- *H* **d.** Write $P \rightarrow Q$ using Peirce arrows only.
- **e.** Write $P \leftrightarrow Q$ using Peirce arrows only.

## ANSWERS FOR TEST YOURSELF

**1.** the output signal(s) that correspond to all possible combinations of input signals to the circuit **2.** a Boolean expression that represents the input signals as variables and indicates the successive actions of the logic gates on the input signals **3.** outputs a 1 for exactly one particular combination of input signals and outputs 0's for all other combinations **4.** they have the same input/output table **5.** NOT; AND **6.** NOT; OR

# 2.5 Application: Number Systems and Circuits for Addition

*Counting in binary is just like counting in decimal if you are all thumbs.* —Glaser and Way

In elementary school, you learned the meaning of decimal notation: that to interpret a string of decimal digits as a number, you mentally multiply each digit by its place value. For instance, 5,049 has a 5 in the thousands place, a 0 in the hundreds place, a 4 in the tens place, and a 9 in the ones place. Thus

$$5{,}049 = 5\cdot(1{,}000) + 0\cdot(100) + 4\cdot(10) + 9\cdot(1).$$

Using exponential notation, this equation can be rewritten as

$$5{,}049 = 5\cdot 10^3 + 0\cdot 10^2 + 4\cdot 10^1 + 9\cdot 10^0.$$

More generally, decimal notation is based on the fact that any positive integer can be written uniquely as a sum of products of the form

$$d\cdot 10^n,$$

where each $n$ is a nonnegative integer and each $d$ is one of the decimal digits 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. The word *decimal* comes from the Latin root *deci*, meaning "ten." Decimal (or base 10) notation expresses a number as a string of digits in which each digit's position indicates the power of 10 by which it is multiplied. The right-most position is the ones place (or $10^0$ place), to the left of that is the tens place (or $10^1$ place), to the left of that is the hundreds place (or $10^2$ place), and so forth, as illustrated below.

| **Place** | $10^3$<br>thousands | $10^2$<br>hundreds | $10^1$<br>tens | $10^0$<br>ones |
|---|---|---|---|---|
| **Decimal Digit** | 5 | 0 | 4 | 9 |

### *Binary Representation of Numbers*

There is nothing sacred about the number 10; we use 10 as a base for our usual number system because we happen to have ten fingers. In fact, any integer greater than 1 can serve as a base for a number system. In computer science, **base 2 notation**, or **binary notation**, is of special importance because the signals used in modern electronics are always in one of only two states. (The Latin root *bi* means "two.")

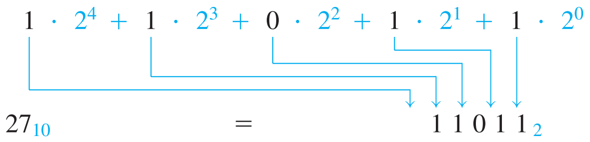
In Section 5.4, we show that any integer can be represented uniquely as a sum of products of the form

$$d \cdot 2^n,$$

where each $n$ is an integer and each $d$ is one of the binary digits (or bits) 0 or 1. For example,

$$\begin{aligned} 27 &= 16 + 8 + 2 + 1 \\ &= 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0. \end{aligned}$$

In binary notation, as in decimal notation, we write just the binary digits, and not the powers of the base. In binary notation, then,



where the subscripts indicate the base, whether 10 or 2, in which the number is written. The places in binary notation correspond to the various powers of 2. The right-most position is the ones place (or $2^0$ place), to the left of that is the twos place (or $2^1$ place), to the left of that is the fours place (or $2^2$ place), and so forth, as illustrated below.

| **Place** | $2^4$<br>sixteens | $2^3$<br>eights | $2^2$<br>fours | $2^1$<br>twos | $2^0$<br>ones |
|---|---|---|---|---|---|
| **Binary Digit** | 1 | 1 | 0 | 1 | 1 |

As in the decimal notation, leading zeros may be added or dropped as desired. For example,

$$003_{10} = 3_{10} = 1 \cdot 2^1 + 1 \cdot 2^0 = 11_2 = 011_2.$$

## Example 2.5.1 Binary Notation for Integers from 1 to 9

Derive the binary notation for the integers from 1 to 9.

**Solution**

$$\begin{aligned}
1_{10} &= & 1 \cdot 2^0 &= 1_2 \\
2_{10} &= & 1 \cdot 2^1 + 0 \cdot 2^0 &= 10_2 \\
3_{10} &= & 1 \cdot 2^1 + 1 \cdot 2^0 &= 11_2 \\
4_{10} &= & 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 &= 100_2 \\
5_{10} &= & 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 &= 101_2 \\
6_{10} &= & 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 &= 110_2 \\
7_{10} &= & 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 &= 111_2 \\
8_{10} &= & 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 &= 1000_2 \\
9_{10} &= & 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 &= 1001_2
\end{aligned}$$

■

A list of powers of 2 is useful for doing binary-to-decimal and decimal-to-binary conversions. See Table 2.5.1.

**TABLE 2.5.1** Powers of 2

| **Power of 2** | $2^{10}$ | $2^9$ | $2^8$ | $2^7$ | $2^6$ | $2^5$ | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Decimal Form** | 1024 | 512 | 256 | 128 | 64 | 32 | 16 | 8 | 4 | 2 | 1 |

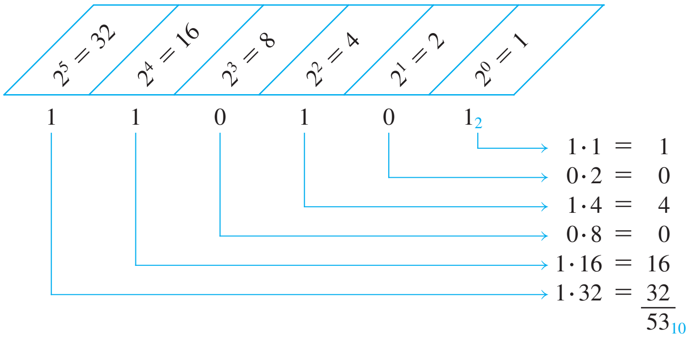
## **Example 2.5.2** Converting a Binary to a Decimal Number

Represent $110101_2$ in decimal notation.

**Solution**

$$\begin{aligned} 110101_2 &= 1\cdot 2^5 + 1\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0 \\ &= 32 + 16 + 4 + 1 \\ &= 53_{10} \end{aligned}$$

Alternatively, the schema below may be used.



■

## **Example 2.5.3** Converting a Decimal to a Binary Number

Represent 209 in binary notation.

**Solution** Use Table 2.5.1 to write 209 as a sum of powers of 2, starting with the highest power of 2 that is less than 209 and continuing to lower powers.

Since 209 is between 128 and 256, the highest power of 2 that is less than 209 is 128. Hence

$$209_{10} = 128 + \text{a smaller number}.$$

Now $209 - 128 = 81$, and 81 is between 64 and 128, so the highest power of 2 that is less than 81 is 64. Hence

$$209_{10} = 128 + 64 + \text{a smaller number}.$$

Continuing in this way, you obtain

$$\begin{aligned} 209_{10} &= 128 + 64 + 16 + 1 \\ &= 1\cdot 2^7 + 1\cdot 2^6 + 0\cdot 2^5 + 1\cdot 2^4 + 0\cdot 2^3 + 0\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0. \end{aligned}$$

For each power of 2 that occurs in the sum, there is a 1 in the corresponding position of the binary number. For each power of 2 that is missing from the sum, there is a 0 in the corresponding position of the binary number. Thus

$$209_{10} = 11010001_2$$

■

Another procedure for converting from decimal to binary notation is discussed in Section 5.1.

**Caution!** Do not read $10_2$ as "ten"; it is the number two. Read $10_2$ as "one oh base two."

## Binary Addition and Subtraction

The computational methods of binary arithmetic are analogous to those of decimal arithmetic. In binary arithmetic the number 2 (which equals $10_2$ in binary notation) plays a role similar to that of the number 10 in decimal arithmetic.

### Example 2.5.4 Addition in Binary Notation

Add $1101_2$ and $111_2$ using binary notation.

**Solution** Because $2_{10} = 10_2$ and $1_{10} = 1_2$, the translation of $1_{10} + 1_{10} = 2_{10}$ to binary notation is

$$\begin{array}{r} 1_2 \\ +\ 1_2 \\ \hline 10_2 \end{array}$$

It follows that adding two 1's together results in a carry of 1 when binary notation is used. Adding three 1's together also results in a carry of 1 since $3_{10} = 11_2$ ("one one base two").

$$\begin{array}{r} 1_2 \\ +\ 1_2 \\ +\ 1_2 \\ \hline 11_2 \end{array}$$

Thus the addition can be performed as follows:

$$\begin{array}{cccccc} & & {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & & \leftarrow \text{carry row} \\ & & 1 & 1 & 0 & 1_2 & \\ + & & & 1 & 1 & 1_2 & \\ \hline & 1 & 0 & 1 & 0 & 0_2 & \end{array}$$

■

### Example 2.5.5 Subtraction in Binary Notation

Subtract $1011_2$ from $11000_2$ using binary notation.

**Solution** In decimal subtraction the fact that $10_{10} - 1_{10} = 9_{10}$ is used to borrow across several columns. For example, consider the following:

$$\begin{array}{cccccc} & & 9 & 9 & & \\ & & & \cancel{1} & 1 & \leftarrow \text{borrow row} \\ & \cancel{1} & \cancel{0} & \cancel{0} & 0_{10} & \\ - & & & 5 & 8_{10} & \\ \hline & & 9 & 4 & 2_{10} & \end{array}$$

In binary subtraction it may also be necessary to borrow across more than one column. But when you borrow a $1_2$ from $10_2$, what remains is $1_2$.

$$\begin{array}{r} 10_2 \\ -\ 1_2 \\ \hline 1_2 \end{array}$$

Thus the subtraction can be performed as follows:

$$\begin{array}{ccccccc} & & 0 & 1 & 1 & & \\ & & & \cancel{1} & \cancel{1} & 1 & \leftarrow \text{borrow row} \\ & 1 & \cancel{1} & \cancel{0} & \cancel{0} & 0_2 & \\ - & & 1 & 0 & 1 & 1_2 & \\ \hline & & 1 & 1 & 0 & 1_2 & \end{array}$$

■

## Circuits for Computer Addition

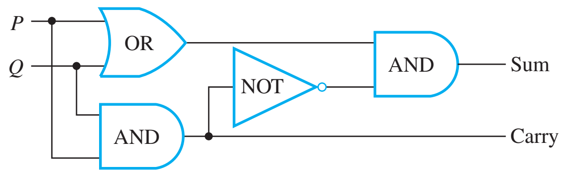
Consider the question of designing a circuit to produce the sum of two binary digits $P$ and $Q$. Both $P$ and $Q$ can be either 0 or 1. And the following facts are known:

$$\begin{aligned} 1_2 + 1_2 & && = 10_2, \\ 1_2 + 0_2 &= 1_2 &&= 01_2, \\ 0_2 + 1_2 &= 1_2 &&= 01_2, \\ 0_2 + 0_2 &= 0_2 &&= 00_2. \end{aligned}$$

It follows that the circuit must have two outputs—one for the left binary digit (this is called the **carry**) and one for the right binary digit (this is called the **sum**). The carry output is 1 if both $P$ and $Q$ are 1; it is 0 otherwise. Thus the carry can be produced using the AND-gate circuit that corresponds to the Boolean expression $P \wedge Q$. The sum output is 1 if either $P$ or $Q$, but not both, is 1. The sum can, therefore, be produced using a circuit that corresponds to the Boolean expression for *exclusive or*: $(P \vee Q) \wedge \sim(P \wedge Q)$. (See Example 2.4.3(a).) Hence, a circuit to add two binary digits $P$ and $Q$ can be constructed as in Figure 2.5.1. This circuit is called a **half-adder**.

**HALF-ADDER**

**Circuit**



**Input/OutputTable**

| ***P*** | ***Q*** | **Carry** | **Sum** |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 |

**FIGURE 2.5.1** Circuit to Add $P + Q$, Where $P$ and $Q$ Are Binary Digits

Now consider the question of how to construct a circuit to add two binary integers, each with more than one digit. Because the addition of two binary digits may result in a carry to the next column to the left, it may be necessary to add three binary digits at certain points. In the following example, the sum in the right column is the sum of two binary digits, and, because of the carry, the sum in the left column is the sum of three binary digits.

$$\begin{array}{rccl} & & 1 & \leftarrow \text{carry row} \\ & & 1 & 1_2 \\ + & & 1 & 1_2 \\ \hline & 1 & 1 & 0_2 \end{array}$$

Thus, in order to construct a circuit that will add multidigit binary numbers, it is necessary to incorporate a circuit that will compute the sum of three binary digits. Such a circuit is called a **full-adder**. Consider a general addition of three binary digits $P$, $Q$, and $R$ that results in a carry (or left-most digit) $C$ and a sum (or right-most digit) $S$.

$$\begin{array}{r} P \\ +\ Q \\ +\ R \\ \hline CS \end{array}$$

The operation of the full-adder is based on the fact that addition is a binary operation: Only two numbers can be added at one time. Thus $P$ is first added to $Q$ and then the result is added to $R$. For instance, consider the following addition:

$$\begin{array}{r} \left.\begin{array}{r}1_2\\ +\ 0_2\end{array}\right\} 1_2+0_2=01_2 \\ +\ 1_2 \\ \hline 10_2 \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} 1_2+1_2=10_2$$

The process illustrated here can be broken down into steps that use half-adder circuits.

**Step 1:** Add $P$ and $Q$ using a half-adder to obtain a binary number with two digits.

$$\begin{array}{r} P \\ +\quad Q \\ \hline C_1S_1 \end{array}$$

**Step 2:** Add $R$ to the sum $C_1$ $S_1$ of $P$ and $Q$.

$$\begin{array}{r} C_1S_1 \\ +\quad R \\ \hline \end{array}$$

To do this, proceed as follows:

**Step 2a:** Add $R$ to $S_1$ using a half-adder to obtain the two-digit number $C_2S$.

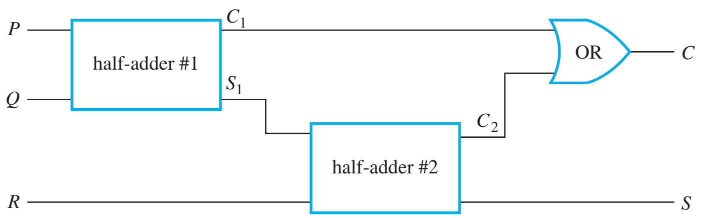
$$\begin{array}{r} S_1 \\ +\quad R \\ \hline C_2S \end{array}$$

Then $S$ is the right-most digit of the entire sum of $P$, $Q$, and $R$.

**Step 2b:** Determine the left-most digit, $C$, of the entire sum as follows: First note that it is impossible for both $C_1$ and $C_2$ to be 1's. For if $C_1 = 1$, then $P$ and $Q$ are both 1, and so $S_1 = 0$. Consequently, the addition of $S_1$ and $R$ gives a binary number $C_2S_1$ where $C_2 = 0$. Next observe that $C$ will be a 1 in the case that the addition of $P$ and $Q$ gives a carry of 1 or in the case that the addition of $S_1$ (the right-most digit of $P + Q$) and $R$ gives a carry of 1. In other words, $C = 1$ if, and only if, $C_1 = 1$ or $C_2 = 1$. It follows that the circuit shown in Figure 2.5.2 will compute the sum of three binary digits.

**FULL-ADDER**

**Circuit**



**Input/Output Table**

| P | Q | R | C | S |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 |

FIGURE 2.5.2 Circuit to Add $P + Q + R$, Where $P$, $Q$, and $R$ Are Binary Digits

Two full-adders and one half-adder can be used together to build a circuit that will add two three-digit binary numbers $PQR$ and $STU$ to obtain the sum $WXYZ$. This is illustrated in Figure 2.5.3. Such a circuit is called a **parallel adder**. Parallel adders can be constructed to add binary numbers of any finite length.

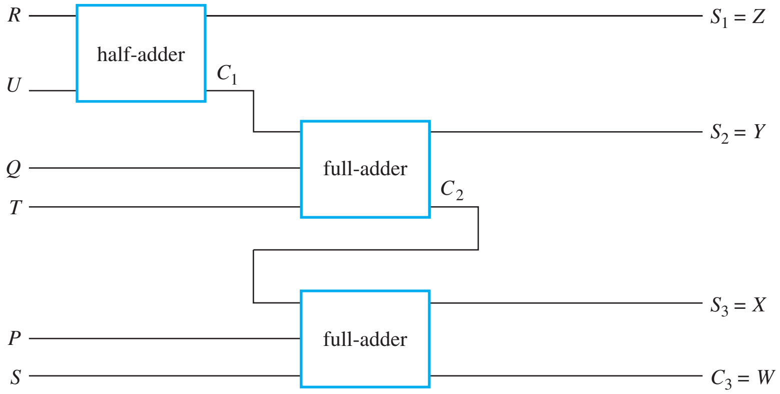


**FIGURE 2.5.3** A Parallel Adder to Add $PQR$ and $STU$ to Obtain $WXYZ$

## *Two's Complements and the Computer Representation of Signed Integers*

Typically a fixed number of bits is used to represent integers on a computer. One way to do this is to select a particular bit, normally the left-most, to indicate the sign of the integer, and to use the remaining bits for its absolute value in binary notation. The problem with this approach is that the procedures for adding the resulting numbers are somewhat complicated and the representation of 0 is not unique. A more common approach is to use "two's complements," which makes it possible to add integers quite easily and results in a unique representation for 0. Bit lengths of 64 and (sometimes) 32 are most often used in practice, but, for simplicity and because the principles are the same for all bit lengths, this discussion will focus on a bit length of 8.

We will show how to use eight bits to represent the 256 integers from −128 through 127 and how to perform additions and subtractions within this system of numbers. When the more realistic 32-bit two's complements system is used, more than 4 billion integers can be represented.

**Definition**

**The 8-bit two's complement** for an integer $a$ between $-128$ and 127 is the 8-bit binary representation for $\begin{cases} a & \text{if } a \geq 0 \\ 2^8 - |a| & \text{if } a < 0. \end{cases}$

Thus the 8-bit representation for a nonnegative integer is the same as its 8-bit binary representation. As a concrete example for the negative integer −46, observe that

$$(2^8 - |-46|)_{10} = (256 - 46)_{10} = 210_{10} = (128 + 64 + 16 + 2)_{10} = 11010010_2,.$$

and so the 8-bit two's complement for −46 is 11010010.

For negative integers, however, there is a more convenient way to compute two's complements, which involves less arithmetic than applying the definition directly.

### The 8-Bit Two's Complement for a Negative Integer

The 8-bit two's complement for a negative integer $a$ that is at least $-128$ can be obtained as follows:

- Write the 8-bit binary representation for $|a|$.
- Switch all the 1's to 0's and all the 0's to 1's. (This is called flipping, or complementing, the bits.)
- Add 1 in binary notation.

## Example 2.5.6 Finding a Two's Complement

Use the method described above to find the 8-bit two's complement for $-46$.

**Solution** Write the 8-bit binary representation for $|-46|$ (=46), switch all the 1's to 0's and all the 0's to 1's, and then add 1.

$$|-46|_{10} = 46_{10} = (32+8+4+2)_{10} = 00101110_2 \xrightarrow{\text{flip the bits}} 11010001 \xrightarrow{\text{add 1}} 11010010.$$

Note that this is the same result as was obtained directly from the definition. ■

The fact that the method for finding 8-bit two's complements works in general depends on the following facts:

1. The binary representation of $2^8 - 1$ is $11111111_2$.
2. Subtracting an 8-bit binary number $a$ from $11111111_2$ switches all the 1's to 0's and all the 0's to 1's.
3. $2^8 - |a| = [(2^8 - 1) - |a|] + 1$ for any number $a$.

Here is how the facts are used when $a = -46$:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | ↔ $2^8 - 1$ |
| | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | ↔ $\lvert -46 \rvert$ |
| 0's and 1's are ↗<br>switched ↘ | | | | | | | | | |
| | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | ↔ $(2^8 - 1) - \lvert -46 \rvert$ |
| 1 is added + | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | ↔ $+1$ |
| | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | ↔ $2^8 - \lvert -46 \rvert$ |

Because 127 is the largest integer represented in the 8-bit two's complement system and because $127_{10} = 01111111_2$, all the 8-bit two's complements for nonnegative integers have a leading bit of 0. Moreover, because the bits are switched, the leading bit for all the negative integers is 1. Table 2.5.2 illustrates the 8-bit two's complement representations for the integers from $-128$ through 127.

**TABLE 2.5.2**

| Integer | 8-Bit Two's Complement | Decimal Form of Two's Complement for Negative Integers |
|---|---|---|
| 127 | 01111111 | |
| 126 | 01111110 | |
| ⋮ | ⋮ | |
| 2 | 00000010 | |
| 1 | 00000001 | |
| 0 | 00000000 | |
| $-1$ | 11111111 | $2^8 - 1$ |
| $-2$ | 11111110 | $2^8 - 2$ |
| $-3$ | 11111101 | $2^8 - 3$ |
| ⋮ | ⋮ | ⋮ |
| $-127$ | 10000001 | $2^8 - 127$ |
| $-128$ | 10000000 | $2^8 - 128$ |

Observe that if the two's complement procedure is used on 11010010, which is the two's complement for $-46$, the result is

$$1101\,0010 \xrightarrow{\text{flip the bits}} 0010\,1101 \xrightarrow{\text{add 1}} 0010\,1110,$$

which is the two's complement for 46. In general, if the two's complement procedure is applied to a positive or negative integer in two's complement form, the result is the negative (or opposite) of that integer. The only exception is the number $-128$. (See exercise 37a.)

To find the decimal representation of the negative integer with a given 8-bit two's complement:

- Apply the two's complement procedure to the given two's complement.
- Write the decimal equivalent of the result.

**Example 2.5.7**

## Finding a Number with a Given Two's Complement

What is the decimal representation for the integer with two's complement 10101001?

**Solution** Since the left-most digit is 1, the integer is negative. Applying the two's complement procedure gives the following result:

$$1010\,1001 \xrightarrow{\text{flip the bits}} 0101\,0110 \xrightarrow{\text{add 1}} 0101\,0111_2$$
$$= (64 + 16 + 4 + 2 + 1)_{10} = 87_{10} = |-87|_{10}.$$

So the answer is $-87$. You can check its correctness by deriving the two's complement of $-87$ directly from the definition:

$$(2^8 - |-87|)_{10} = (256 - 87)_{10} = 169_{10} = (128 + 32 + 8 + 1)_{10} = 10101001_2. \quad ■$$

## *Addition and Subtraction with Integers in Two's Complement Form*

The main advantage of a two's complement representation for integers is that the same computer circuits used to add nonnegative integers in binary notation can be used for both additions and subtractions of integers in a two's complement system of numeration. First note that because of the algebraic identity

$$a - b = a + (-b) \text{ for all real numbers,}$$

any subtraction problem can be changed into an addition one. For example, suppose you want to compute $78 - 46$. This equals $78 + (-46)$, which should give an answer of 32. To see what happens when you add the numbers in their two's complement forms, observe that the 8-bit two's complement for 78 is the same as the ordinary binary representation for 78, which is 01001110 because $78 = 64 + 8 + 4 + 2$, and, as previously shown, the 8-bit two's complement for $-46$ is 11010010. Adding the numbers using binary addition gives the following:

$$\begin{array}{c|c|c|c|c|c|c|c|c|l}
\cline{2-9}
 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & \leftrightarrow 78 \\
\cline{2-9}
\multicolumn{10}{c}{} \\
\cline{2-9}
+ & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \leftrightarrow -46 \\
\cline{2-9}
\multicolumn{10}{c}{} \\
\cline{1-9}
\multicolumn{10}{c}{} \\
\cline{2-9}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \leftrightarrow 32? \\
\cline{2-9}
\end{array}$$

The result has a carry bit of 1 in the ninth, or $2^8$th, position, but if you discard it, you obtain 00100000, which *is* the correct answer in 8-bit two's complement form because, since $32 = 2^8$,

$$32_{10} = 00100000_2.$$

In general, if you add numbers in 8-bit two's complement form and get a carry bit of 1 in the ninth, or $2^8$th position, you should discard it. Using this procedure is equivalent to reducing the sum of the numbers "modulo $2^8$," and it gives results that are correct in ordinary decimal arithmetic as long as the sum of the two numbers is within the fixed-bit-length system of integer representations you are using, in this case those between $-128$ and 127. The fact that this method produces correct results follows from general properties of modular arithmetic, which is discussed at length in Section 8.4.

### General Procedure for Using 8-Bit Two's Complements to Add Two Integers

To add two integers in the range $-128$ through 127 whose sum is also in the range $-128$ through 127:

- Convert both integers to their 8-bit two's complement representations.
- Add the resulting integers using ordinary binary addition, discarding any carry bit of 1 that may occur in the $2^8$th position.
- Convert the result back to decimal form.

When integers are restricted to the range $-128$ through 127, you can easily imagine adding two integers and obtaining a sum outside the range. For instance,

$(-87) + (-46) = -133$, which is less than $-128$ and, therefore, requires more than eight bits for its representation. Because this result is outside the 8-bit fixed-length register system imposed by the architecture of the computer, it is often labeled "overflow error." In the more realistic environment where integers are represented using 64 bits, they can range from less than $-10^{19}$ to more than $10^{19}$. So a vast number of integer calculations can be made without producing overflow error. And even if a 32-bit fixed integer length is used, nearly 4 billion integers are represented within the system.

Detecting overflow error turns out to be quite simple. The 8-bit two's complement sum of two integers will be outside the range from $-128$ through 127 if, and only if, the integers are both positive and the sum computed using 8-bit two's complements is negative, or if the integers are both negative and the sum computed using 8-bit two's complement is positive. To see a concrete example for how this works, consider trying to add $(-87)$ and $(-46)$. Here is what you obtain:

$$\begin{array}{c|c|c|c|c|c|c|c|c|l}
\cline{2-9} & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & \leftrightarrow -87 \\
\cline{2-9} + & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \leftrightarrow -46 \\
\hline 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & \\
\cline{2-9}
\end{array}$$

When you discard the 1 in the $2^8$th position, you find that the leading digit of the result is 0, which would mean that the number with the two's complement representation for the sum of two negative numbers would be positive. So the computer signals an overflow error.*

## Hexadecimal Notation

It should now be obvious that numbers written in binary notation take up much more space than numbers written in decimal notation. Yet many aspects of computer operation can best be analyzed using binary numbers. **Hexadecimal notation** is even more compact than decimal notation, and it is much easier to convert back and forth between hexadecimal and binary notation than it is between binary and decimal notation. The word *hexadecimal* comes from the Greek root *hex*-, meaning "six," and the Latin root *deci*-, meaning "ten." Hence *hexadecimal* refers to "sixteen," and hexadecimal notation is also called **base 16 notation**. Hexadecimal notation is based on the fact that any integer can be uniquely expressed as a sum of numbers of the form

$$d \cdot 16^n,$$

where each $n$ is a nonnegative integer and each $d$ is one of the integers from 0 to 15. In order to avoid ambiguity, each hexadecimal digit must be represented by a single symbol. The integers 10 through 15 are represented by the symbols A, B, C, D, E, and F. The 16 hexadecimal digits are shown in Table 2.5.3, together with their decimal equivalents and, for future reference, their 4-bit binary equivalents.

---

*If the carry bit had not been discarded and if the resulting 9 bits could be processed using a "9-bit two's complement conversion procedure," the result of 101111011 would convert to $-133$, which is the correct answer. However, the computer signals an error because $-133$ is not representable within its 8-bit two's complement system.

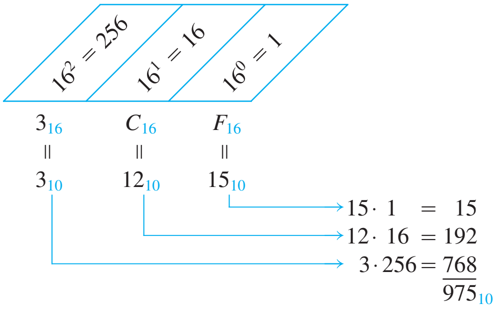
**TABLE 2.5.3**

| Decimal | Hexadecimal | 4-Bit Binary Equivalent |
|---|---|---|
| 0 | 0 | 0000 |
| 1 | 1 | 0001 |
| 2 | 2 | 0010 |
| 3 | 3 | 0011 |
| 4 | 4 | 0100 |
| 5 | 5 | 0101 |
| 6 | 6 | 0110 |
| 7 | 7 | 0111 |
| 8 | 8 | 1000 |
| 9 | 9 | 1001 |
| 10 | A | 1010 |
| 11 | B | 1011 |
| 12 | C | 1100 |
| 13 | D | 1101 |
| 14 | E | 1110 |
| 15 | F | 1111 |

### **Example 2.5.8** Converting from Hexadecimal to Decimal Notation

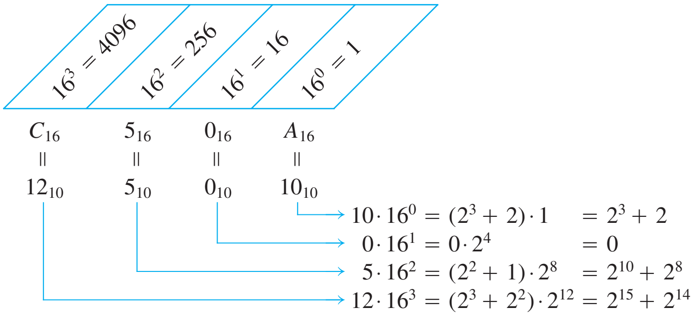
Convert $3CF_{16}$ to decimal notation.

**Solution** A schema similar to the one introduced in Example 2.5.2 can be used here.



So $3CF_{16} = 975_{10}$. ■

Now consider how to convert from hexadecimal to binary notation. In the example below the numbers are rewritten using powers of 2, and the laws of exponents are applied. The result suggests a general procedure.



since $10 = 2^3 + 2$

since $16^1 = 2^4$

since $5 = 2^2 + 1$, $16^2 = (2^4)^2 = 2^8$ and $2^2 \cdot 2^8 = 2^{10}$

since $12 = 2^3 + 2^2$, $16^2 = (2^4)^3 = 2^{12}$, $2^3 \cdot 2^{12} = 2^{15}$, and $2^2 \cdot 2^{12} = 2^{14}$

But

$$\begin{aligned}(2^{15}+2^{14})+(2^{10}+2^{8})+0+(2^{3}+2)\\ &= 1100\,0000\,0000\,0000_2 + 0101\,0000\,0000_2\\ &\quad + 0000\,0000_2 + 1010_2\end{aligned}$$

by the rules for writing binary numbers.

So

$$C50A_{16} = \underbrace{1100}_{C_{16}}\;\underbrace{0101}_{5_{16}}\;\underbrace{0000}_{0_{16}}\;\underbrace{1010_2}_{A_{16}}$$

by the rules for adding binary numbers.

The procedure illustrated in this example can be generalized. In fact, the following sequence of steps will always give the correct answer.

> To convert an integer from hexadecimal to binary notation:
>
> - Write each hexadecimal digit of the integer in 4-bit binary notation.
> - Juxtapose the results.

**Example 2.5.9**

### Converting from Hexadecimal to Binary Notation

Convert $B09F_{16}$ to binary notation.

**Solution** $B_{16} = 11_{10} = 1011_2$, $0_{16} = 0_{10} = 0000_2$, $9_{16} = 9_{10} = 1001_2$, and $F_{16} = 15_{10} = 1111_2$. Consequently,

$$\begin{array}{cccc} \text{B} & 0 & 9 & \text{F} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1011 & 0000 & 1001 & 1111 \end{array}$$

and the answer is $1011000010011111_2$. ■

To convert integers written in binary notation into hexadecimal notation, reverse the steps of the previous procedure. Note that the commonly used computer representation for integers uses 32 bits. When these numbers are written in hexadecimal notation only eight characters are needed.

> To convert an integer from binary to hexadecimal notation:
>
> - Group the digits of the binary number into sets of four, starting from the right and adding leading zeros as needed.
> - Convert the binary numbers in each set of four into hexadecimal digits. Juxtapose those hexadecimal digits.

**Example 2.5.10**

### Converting from Binary to Hexadecimal Notation

Convert $100110110101001_2$ to hexadecimal notation.

**Solution** First group the binary digits in sets of four, working from right to left and adding leading 0's if necessary.

$$0100 \;\; 1101 \;\; 1010 \;\; 1001.$$

Convert each group of four binary digits into a hexadecimal digit.

$$\begin{array}{cccc} 0100 & 1101 & 1010 & 1001 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 4 & D & A & 9 \end{array}$$

Then juxtapose the hexadecimal digits.

$$4DA9_{16}$$ ■

**Example 2.5.11**

### Reading a Memory Dump

The smallest addressable memory unit on most computers is one byte, or eight bits. In some debugging operations a dump is made of memory contents; that is, the contents of each memory location are displayed or printed out in order. To save space and make the output easier on the eye, the hexadecimal versions of the memory contents are given, rather than the binary versions. Suppose, for example, that a segment of the memory dump looks like

A3 BB 59 2E.

What is the actual content of the four memory locations?

**Solution**

$$\begin{aligned} A3_{16} &= 10100011_2 \\ BB_{16} &= 10111011_2 \\ 59_{16} &= 01011001_2 \\ 2E_{16} &= 00101110_2 \end{aligned}$$ ■

## TEST YOURSELF

1. To represent a nonnegative integer in binary notation means to write it as a sum of products of the form ______, where ______.
2. To add integers in binary notation, you use the facts that $1_2 + 1_2 =$ ______ and $1_2 + 1_2 + 1_2 =$ ______.
3. To subtract integers in binary notation, you use the facts that $10_2 - 1_2 =$ ______ and $11_2 - 1_2 =$ ______.
4. A half-adder is a digital logic circuit that ______, and a full-adder is a digital logic circuit that ______.
5. If $a$ is an integer with $-128 \le a \le 127$, the 8-bit two's complement of $a$ is ______ if $a \ge 0$ and is ______ if $a < 0$.
6. To find the 8-bit two's complement of a negative integer $a$ that is at least $-128$, you ______, ______, and ______.
7. To add two integers in the range $-128$ through 127 whose sum is also in the range $-128$ through 127, you ______, ______, ______, and ______.
8. To represent a nonnegative integer in hexadecimal notation means to write it as a sum of products of the form ______, where ______.
9. To convert a nonnegative integer from hexadecimal to binary notation, you ______ and ______.

## EXERCISE SET 2.5

Represent the decimal integers in 1–6 in binary notation.

1. 19
2. 55
3. 287
4. 458
5. 1609
6. 1424

Represent the integers in 7–12 in decimal notation.

7. $1110_2$
8. $10111_2$
9. $110110_2$
10. $1100101_2$
11. $1000111_2$
12. $1011011_2$

Perform the arithmetic in 13–20 using binary notation.

**13.** $$\begin{array}{r} 1011_2 \\ +\ \ 101_2 \\ \hline \end{array}$$

**14.** $$\begin{array}{r} 1001_2 \\ +1011_2 \\ \hline \end{array}$$

**15.** $$\begin{array}{r} 101101_2 \\ +\ \ 11101_2 \\ \hline \end{array}$$

**16.** $$\begin{array}{r} 110111011_2 \\ +1001011010_2 \\ \hline \end{array}$$

**17.** $$\begin{array}{r} 10100_2 \\ -\ \ 1101_2 \\ \hline \end{array}$$

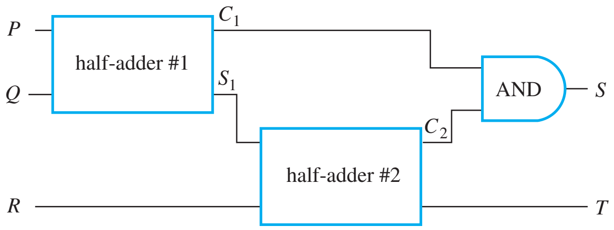
**18.** $$\begin{array}{r} 11010_2 \\ -\ \ 1101_2 \\ \hline \end{array}$$

**19.** $$\begin{array}{r} 101101_2 \\ -\ \ 10011_2 \\ \hline \end{array}$$

**20.** $$\begin{array}{r} 1010100_2 \\ -\ \ \ 10111_2 \\ \hline \end{array}$$

**21.** Give the output signals $S$ and $T$ for the circuit shown below if the input signals $P$, $Q$, and $R$ are as specified. Note that this is *not* the circuit for a full-adder.

**a.** $P = 1, Q = 1, R = 1$
**b.** $P = 0, Q = 1, R = 0$
**c.** $P = 1, Q = 0, R = 1$



**22.** Add $11111111_2 + 1_2$ and convert the result to decimal notation, to verify that $11111111_2 = (2^8 - 1)_{10}$.

Find the 8-bit two's complements for the integers in 23–26.

**23.** −23 **24.** −67 **25.** −4 **26.** −115

Find the decimal representations for the integers with the 8-bit two's complements given in 27–30.

**27.** 11010011 **28.** 10011001

**29.** 11110010 **30.** 10111010

Use 8-bit two's complements to compute the sums in 31–36.

**31.** $57 + (-118)$ **32.** $62 + (-18)$

**33.** $(-6) + (-73)$ **34.** $89 + (-55)$

**35.** $(-15) + (-46)$ **36.** $123 + (-94)$

**37. a.** Show that when you apply the 8-bit two's complement procedure to the 8-bit two's complement for −128, you get the 8-bit two's complement for −128.

*__b.__ Show that if $a$, $b$, and $a + b$ are integers in the range 1 through 128, then

$$(2^8 - a) + (2^8 - b) = (2^8 - (a + b)) + 2^8 \geq 2^8 + 2^7.$$

Explain why it follows that if integers $a$, $b$, and $a + b$ are all in the range 1 through 128, then the 8-bit two's complement of $(-a) + (-b)$ is a negative number.

Convert the integers in 38–40 from hexadecimal to decimal notation.

**38.** $\text{A2BC}_{16}$ **39.** $\text{E0D}_{16}$ **40.** $\text{39EB}_{16}$

Convert the integers in 41–43 from hexadecimal to binary notation.

**41.** $\text{1C0ABE}_{16}$ **42.** $\text{B53DF8}_{16}$ **43.** $\text{4ADF83}_{16}$

Convert the integers in 44–46 from binary to hexadecimal notation.

**44.** $00101110_2$ **45.** $1011011111000101_2$

**46.** $11001001011100_2$

**47.** **Octal Notation:** In addition to binary and hexadecimal, computer scientists also use *octal notation* (base 8) to represent numbers. Octal notation is based on the fact that any integer can be uniquely represented as a sum of numbers of the form $d \cdot 8^n$, where each $n$ is a nonnegative integer and each $d$ is one of the integers from 0 to 7. Thus, for example, $5073_8 = 5 \cdot 8^3 + 0 \cdot 8^2 + 7 \cdot 8^1 + 3 \cdot 8^0 = 2619_{10}$.

**a.** Convert $61502_8$ to decimal notation.
**b.** Convert $20763_8$ to decimal notation.
**c.** Describe methods for converting integers from octal to binary notation and the reverse that are similar to the methods used in Examples 2.5.9 and 2.5.10 for converting back and forth from hexadecimal to binary notation. Give examples showing that these methods result in correct answers.

## ANSWERS FOR TEST YOURSELF

**1.** $d \cdot 2^n$; $d = 0$ or $d = 1$, and $n$ is a nonnegative integer **2.** $10_2$; $11_2$ **3.** $1_2$; $10_2$ **4.** outputs the sum of any two binary digits; outputs the sum of any three binary digits **5.** the 8-bit binary representation of $a$; the 8-bit binary representation of $2^8 - a$ **6.** write the 8-bit binary representation of $a$; flip the bits; add 1 in binary notation **7.** convert both integers to their 8-bit two's complements; add the results using binary notation; truncate any leading 1; convert back to decimal form **8.** $d \cdot 16^n$; $d = 0, 1, 2, \ldots 9, A, B, C, D, E, F$, and $n$ is a nonnegative integer **9.** write each hexadecimal digit in 4-bit binary notation; juxtapose the results

CHAPTER 3

# THE LOGIC OF QUANTIFIED STATEMENTS

In Chapter 2 we discussed the logical analysis of compound statements—those made of simple statements joined by the connectives $\sim$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. Such analysis casts light on many aspects of human reasoning, but it cannot be used to determine validity in the majority of everyday and mathematical situations. For example, the argument

All men are mortal.
Socrates is a man.
∴ Socrates is mortal.

is intuitively perceived as correct. Yet its validity cannot be derived using the methods outlined in Section 2.3. To determine validity in examples like this, it is necessary to separate the statements into parts in much the same way that you separate declarative sentences into subjects and predicates. And you must analyze and understand the special role played by words that denote quantities such as "all" or "some." The symbolic analysis of predicates and quantified statements is called the **predicate calculus**. The symbolic analysis of ordinary compound statements (as outlined in Sections 2.1–2.3) is called the **statement calculus** (or the **propositional calculus**).

## 3.1 Predicates and Quantified Statements I

*. . . it was not till within the last few years that it has been realized how fundamental any and some are to the very nature of mathematics.* —A. N. Whitehead (1861–1947)

As noted in Section 2.1, the sentence "$x^2 + 2 = 11$" is not a statement because it may be either true or false depending on the value of $x$. Similarly, the sentence "$x + y > 0$" is not a statement because its truth value depends on the values of the variables $x$ and $y$.

In grammar, the word *predicate* refers to the part of a sentence that gives information about the subject. In the sentence "James is a student at Bedford College," the word *James* is the subject and the phrase *is a student at Bedford College* is the predicate. The predicate is the part of the sentence from which the subject has been removed.

In logic, predicates can be obtained by removing some or all of the nouns from a statement. For instance, let $P$ stand for "is a student at Bedford College" and let $Q$ stand for "is a student at." Then both $P$ and $Q$ are *predicate symbols*. The sentences "$x$ is a student at Bedford College" and "$x$ is a student at $y$" are symbolized as $P(x)$ and as $Q(x, y)$, respectively, where $x$ and $y$ are *predicate variables* that take values in appropriate sets. When concrete values are substituted in place of predicate variables, a statement results. For simplicity, we define a *predicate* to be a predicate symbol together with suitable predicate variables. In some other treatments of logic, such objects are referred to as **propositional functions** or **open sentences**.

**Definition**

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

**Example 3.1.1**

### Finding Truth Values of a Predicate

Let $P(x)$ be the predicate "$x^2 > x$" with domain the set $\mathbf{R}$ of all real numbers. Write $P(2)$, $P\left(\frac{1}{2}\right)$, and $P\left(-\frac{1}{2}\right)$, and indicate which of these statements are true and which are false.

**Solution**

$$P(2):\quad 2^2 > 2, \quad \text{or} \quad 4 > 2. \quad \text{True.}$$

$$P\left(\frac{1}{2}\right):\quad \left(\frac{1}{2}\right)^2 > \frac{1}{2}, \quad \text{or} \quad \frac{1}{4} > \frac{1}{2}. \quad \text{False.}$$

$$P\left(-\frac{1}{2}\right):\quad \left(-\frac{1}{2}\right)^2 > -\frac{1}{2} \quad \text{or} \quad \frac{1}{4} > -\frac{1}{2}. \quad \text{True.}$$

■

When an element in the domain of the variable of a one-variable predicate is substituted for the variable, the resulting statement is either true or false. The set of all such elements that make the predicate true is called the *truth set* of the predicate.

**Definition**

If $P(x)$ is a predicate and $x$ has domain $D$, the **truth set** of $P(x)$ is the set of all elements of $D$ that make $P(x)$ true when they are substituted for $x$. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

**Note** Recall that we read these symbols as "the set of all $x$ in $D$ such that $P(x)$."

**Example 3.1.2**

### Finding the Truth Set of a Predicate

Let $Q(n)$ be the predicate "$n$ is a factor of 8." Find the truth set of $Q(n)$ if

a. the domain of $n$ is $\mathbf{Z}^+$, the set of all positive integers

b. the domain of $n$ is $\mathbf{Z}$, the set of all integers.

**Solution**

a. The truth set is {1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly.

b. The truth set is {1, 2, 4, 8, −1, −2, −4, −8} because the negative integers −1, −2, −4, and −8 also divide into 8 without leaving a remainder. ■

## *The Universal Quantifier: ∀*

One sure way to change predicates into statements is to assign specific values to all their variables. For example, if $x$ represents the number 35, the sentence "$x$ is (evenly) divisible by 5" is a true statement since $35 = 5 \cdot 7$. Another way to obtain statements from predicates is to add **quantifiers**. Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true. The formal concept of quantifier was introduced into symbolic logic in the late nineteenth century by the American philosopher, logician, and engineer Charles Sanders Peirce and, independently, by the German logician Gottlob Frege.

*Charles Sanders Peirce (1839–1914)*

Fine Art Images/Glow Images

**Note** Think "for every" when you see the symbol $\forall$.

The symbol $\forall$ is called the **universal quantifier**. Depending on the context, it is read as "for every," "for each," "for any," "given any," or "for all." For example, another way to express the sentence "Every human being is mortal" or "All human beings are mortal" is to write

$$\forall \text{ human beings } x,\ x \text{ is mortal,}$$

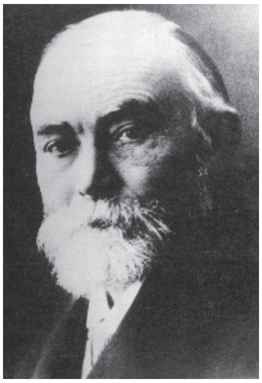
which you would read as "For every human being $x$, $x$ is mortal." If you let $H$ be the set of all human beings, then you can symbolize the statement more formally by writing

$$\forall x \in H,\ x \text{ is mortal.}$$

Think of the symbol $x$ as an individual but generic object, with all the properties shared by every human being but with no other properties. Because $x$ is individual, even if you read $\forall$ as "for all," you should use the singular verb and say, "For all $x$ in $H$, $x$ is mortal" rather than "For all $x$ in $H$, $x$ are mortal."

In a universally quantified sentence the domain of the predicate variable is generally indicated either between the $\forall$ symbol and the variable name (as in $\forall$ human being $x$) or immediately following the variable name (as in $\forall x \in H$). In sentences containing a mixture of symbols and words, the $\forall$ symbol can refer to two or more variables. For instance, you could symbolize "For all real numbers $x$ and $y$, $x + y = y + x$." as "$\forall$ real numbers $x$ and $y$, $x + y = y + x$."*

Sentences that are quantified universally are defined as statements by giving them the truth values specified in the following definition:



*Gottlob Frege (1848–1925)*

Pictorial Press Ltd./Alamy Stock Photo

**Definition**

Let $Q(x)$ be a predicate and $D$ the domain of $x$. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for each individual $x$ in $D$. It is defined to be false if, and only if, $Q(x)$ is false for at least one $x$ in $D$. A value for $x$ for which $Q(x)$ is false is called a **counterexample** to the universal statement.

## Example 3.1.3 Truth and Falsity of Universal Statements

a. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D,\ x^2 \geq x.$$

Write one way to read this statement out loud, and show that it is true.

b. Consider the statement

$$\forall x \in \mathbf{R},\ x^2 \geq x.$$

Find a counterexample to show that this statement is false.

---

*More formal versions of symbolic logic would require a separate $\forall$ for each variable: "$\forall x \in \mathbf{R}(\forall y \in \mathbf{R}(x + y = y + x))$."

**Solution**

a. "For every $x$ in the set $D$, $x^2$ is greater than or equal to $x$." The inequalities below show that "$x^2 \geq x$" is true for each individual $x$ in $D$.

$$1^2 \geq 1, \quad 2^2 \geq 2, \quad 3^2 \geq 3, \quad 4^2 \geq 4, \quad 5^2 \geq 5.$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

b. *Counterexample:* The statement claims that $x^2 \geq x$ for every real number $x$. But when $x = \frac{1}{2}$, for example,

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not\geq \frac{1}{2}.$$

Hence "$\forall x \in \mathbf{R}, x^2 \geq x$" is false. ■

The technique used to show the truth of the universal statement in Example 3.1.3(a) is called the **method of exhaustion**. It consists of showing the truth of the predicate separately for each individual element of the domain. (The idea is to exhaust the possibilities before you exhaust yourself!) This method can, in theory, be used whenever the domain of the predicate variable is finite. In recent years the prevalence of digital computers has greatly increased the convenience of using the method of exhaustion. Computer expert systems, or knowledge-based systems, use this method to arrive at answers to many of the questions posed to them. Because most mathematical sets are infinite, however, the method of exhaustion can rarely be used to derive general mathematical results.

## *The Existential Quantifier: ∃*

The symbol ∃ denotes "there exists" and is called the **existential quantifier**. For example, the sentence "There is a student in Math 140" can be written as

**Note** Think "there exists" when you see the symbol ∃.

$$\exists \text{ a person } p \text{ such that } p \text{ is a student in Math 140,}$$

or, more formally,

$$\exists p \in P \text{ such that } p \text{ is a student in Math 140,}$$

where $P$ is the set of all people. The domain of the predicate variable is generally indicated either between the ∃ symbol and the variable name or immediately following the variable name, and the words *such that* are inserted just before the predicate. Some other expressions that can be used in place of *there exists* are *there is a*, *we can find a*, *there is at least one*, *for some*, and *for at least one*. In a sentence such as "∃ integers $m$ and $n$ such that $m + n = m \cdot n$," the ∃ symbol is understood to refer to both $m$ and $n$.*

Sentences that are quantified existentially are defined as statements by giving them the truth values specified in the following definition.

**Definition**

Let $Q(x)$ be a predicate and $D$ the domain of $x$. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one $x$ in $D$. It is false if, and only if, $Q(x)$ is false for all $x$ in $D$.

---

*In more formal versions of symbolic logic, the words *such that* are not written out (although they are understood) and a separate ∃ symbol is used for each variable: "$\exists m \in \mathbf{Z}\ (\exists n \in \mathbf{Z}(m + n = m \cdot n))$."

### Example 3.1.4 Truth and Falsity of Existential Statements

a. Consider the statement

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$$

Write one way to read this statement out loud, and show that it is true.

b. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

### Solution

a. "There is at least one positive integer $m$ such that $m^2 = m$." Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for a positive integer $m$, and so "$\exists m \in \mathbf{Z}^+$ such that $m^2 = m$" is true.

b. Note that $m^2 = m$ is not true for any integers $m$ from 5 through 8:

$$5^2 = 25 \neq 5, \quad 6^2 = 36 \neq 6, \quad 7^2 = 49 \neq 7, \quad 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false. ■

## Formal vs. Informal Language

It is important to be able to translate from formal to informal language when trying to make sense of mathematical concepts that are new to you. It is equally important to be able to translate from informal to formal language when thinking out a complicated problem.

### Example 3.1.5 Translating from Formal to Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol $\forall$ or $\exists$.

a. $\forall x \in \mathbf{R}, x^2 \geq 0.$

b. $\forall x \in \mathbf{R}, x^2 \neq -1.$

c. $\exists m \in \mathbf{Z}^+$ such that $m^2 = m$.

### Solution

a. Every real number has a nonnegative square.
*Or:* All real numbers have nonnegative squares.
*Or:* Any real number has a nonnegative square.
*Or:* The square of each real number is nonnegative.

b. All real numbers have squares that do not equal $-1$.
*Or:* No real numbers have squares equal to $-1$.
(The words *none are* or *no … are* are equivalent to the words *all are not*.)

**Note** In ordinary English, the fourth statement in part (c) may be taken to mean that there are at least two positive integers equal to their own squares. In mathematics, we understand the last two statements in part (c) to mean the same thing.

c. There is a positive integer whose square is equal to itself.
*Or:* We can find at least one positive integer equal to its own square.
*Or:* Some positive integer equals its own square.
*Or:* Some positive integers equal their own squares. ■

Another way to restate universal and existential statements informally is to place the quantification at the end of the sentence. For instance, instead of saying "For any real number $x$, $x^2$ is nonnegative," you could say "$x^2$ is nonnegative for any real number $x$." In such a case the quantifier is said to "trail" the rest of the sentence.

**Example 3.1.6**

### Trailing Quantifiers

Rewrite the following statements so that the quantifier trails the rest of the sentence.

a. For any integer $n$, $2n$ is even.

b. There exists at least one real number $x$ such that $x^2 \leq 0$.

**Solution**

a. $2n$ is even for any integer $n$.

b. $x^2 \leq 0$ for some real number $x$.
*Or:* $x^2 \leq 0$ for at least one real number $x$. ■

**Example 3.1.7**

### Translating from Informal to Formal Language

**Note** The following two sentences mean the same thing: "All triangles have three sides" and "Every triangle has three sides."

Rewrite each of the following statements formally. Use quantifiers and variables.

a. All triangles have three sides.

b. No dogs have wings.

c. Some programs are structured.

**Solution**

a. $\forall$ triangle $t$, $t$ has three sides.
*Or:* $\forall t \in T$, $t$ has three sides (where $T$ is the set of all triangles).

b. $\forall$ dog $d$, $d$ does not have wings.
*Or:* $\forall d \in D$, $d$ does not have wings (where $D$ is the set of all dogs).

c. $\exists$ a program $p$ such that $p$ is structured.
*Or:* $\exists p \in P$ such that $p$ is structured (where $P$ is the set of all programs). ■

## *Universal Conditional Statements*

A reasonable argument can be made that the most important form of statement in mathematics is the **universal conditional statement**:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Familiarity with statements of this form is essential if you are to learn to speak mathematics.

**Example 3.1.8**

### Writing Universal Conditional Statements Informally

Rewrite the following statement informally, without quantifiers or variables.

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$$

**Solution** If a real number is greater than 2, then its square is greater than 4.

*Or:* Whenever a real number is greater than 2, its square is greater than 4.

*Or:* The square of any real number greater than 2 is greater than 4.

*Or:* The squares of all real numbers greater than 2 are greater than 4. ■

**Example 3.1.9**

### Writing Universal Conditional Statements Formally

Rewrite each of the following statements in the form

$$\forall \underline{\qquad}, \text{ if } \underline{\qquad} \text{ then } \underline{\qquad}.$$

a. If a real number is an integer, then it is a rational number.

b. All bytes have eight bits.

c. No fire trucks are green.

**Solution**

a. $\forall$ real number $x$, if $x$ is an integer, then $x$ is a rational number.
*Or:* $\forall x \in \mathbf{R}$, if $x \in \mathbf{Z}$ then $x \in \mathbf{Q}$.

b. $\forall x$, if $x$ is a byte, then $x$ has eight bits.

c. $\forall x$, if $x$ is a fire truck, then $x$ is not green.

It is common, as in (b) and (c) above, to omit explicit identification of the domain of predicate variables in universal conditional statements. ■

Careful thought about the meaning of universal conditional statements leads to another level of understanding for why the truth table for an if-then statement must be defined as it is. Consider again the statement

$$\forall \text{ real number } x, \text{ if } x > 2 \text{ then } x^2 > 4.$$

Your experience and intuition tell you that this statement is true. But that means that

$$\text{If } x > 2 \text{ then } x^2 > 4$$

must be true for every single real number $x$. Consequently, it must be true even for values of $x$ that make its hypothesis "$x > 2$" false. In particular, both statements

$$\text{If } 1 > 2 \text{ then } 1^2 > 4 \quad \text{and} \quad \text{If } -3 > 2 \text{ then } (-3)^2 > 4$$

must be true. In both cases the hypothesis is false, but in the first case the conclusion "$1^2 > 4$" is false, and in the second case the conclusion "$(-3)^2 > 4$" is true. Hence, if an if-then statement has a false hypothesis, we have to interpret it as true regardless of whether its conclusion is true or false.

Note also that the definition of valid argument is a universal conditional statement:

> For every combination of truth values for the component statements,
> if the premises are all true then the conclusion is also true.

## *Equivalent Forms of Universal and Existential Statements*

Observe that the two statements "$\forall$ real number $x$, if $x$ is an integer then $x$ is rational" and "$\forall$ integer $x$, $x$ is rational" mean the same thing because the set of integers is a subset of the set of real numbers. Both have informal translations "All integers are rational." In fact, a statement of the form

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x)$$

can always be rewritten in the form

$$\forall x \in D, Q(x)$$

by narrowing $U$ to be the subset $D$ consisting of all values of the variable $x$ that make $P(x)$ true. Conversely, a statement of the form

$$\forall x \in D, Q(x)$$

can be rewritten as

$$\forall x, \text{ if } x \text{ is in } D \text{ then } Q(x).$$

**Example 3.1.10** **Equivalent Forms for Universal Statements**

Rewrite the following statement in the two forms "$\forall x$, if ______ then ______" and "$\forall$ ______ $x$, ______": All squares are rectangles.

**Solution** $\forall x$, if $x$ is a square then $x$ is a rectangle.
$\forall$ square $x$, $x$ is a rectangle. ■

Similarly, a statement of the form "$\exists x$ such that $P(x)$ and $Q(x)$" can be rewritten as "$\exists x \in D$ such that $Q(x)$," where $D$ is the set of all $x$ for which $P(x)$ is true.

**Example 3.1.11** **Equivalent Forms for Existential Statements**

A **prime number** is an integer greater than 1 whose only positive integer factors are itself and 1. Consider the statement "There is an integer that is both prime and even." Let Prime($n$) be "$n$ is prime" and Even($n$) be "$n$ is even." Use the notation Prime($n$) and Even($n$) to rewrite this statement in the following two forms:

a. $\exists n$ such that ______ $\wedge$ ______.

b. $\exists$ ______ $n$ such that ______.

**Solution**

a. $\exists n$ such that Prime($n$) $\wedge$ Even($n$).

b. Two answers: $\exists$ a prime number $n$ such that Even($n$).
$\exists$ an even number $n$ such that Prime($n$). ■

## Bound Variables and Scope

Consider the statement "For every integer $x$, $x^2 \geq 0$." First note that you don't have to call the variable $x$. You can use any name for it as long as you do so consistently. For instance, all the following statements have the same meaning:

For every integer $x$, $x^2 \geq 0$. For every integer $n$, $n^2 \geq 0$. For every integer $s$, $s^2 \geq 0$.

In each case the variable simply holds a place for any element in the set of all integers. Each way of writing the statement says that whatever integer you might choose, when you square it the result will be nonnegative. It is important to note, however, that once you finish writing the statement, whatever symbol you chose to use in it can be given an entirely different meaning when used in a different context.

For example, consider the following statements:

(1) For every integer $x$, $x^2 \geq 0$.

(2) There exists a real number $x$ such that $x^3 = 8$.

Statements (1) and (2) both call the variable $x$, but the $x$ in Statement (1) serves a different function from the $x$ in Statement (2). We say that the variable $x$ is **bound** by the quantifier that controls it and that its **scope** begins when the quantifier introduces it and ends at the end of the quantified statement.

The way variables are used in mathematics is similar to the way they are used in computer programming. A variable in a computer program also serves as a placeholder in the sense that it creates a location in computer memory (either actual or virtual) into which its values can be placed. In addition the way it can be bound in a program is similar to the way that a mathematical variable can be bound in a statement. For example, consider the following two examples in Python:

**Program 1**

```
def f():
  X = "Hi"
  print X
def g():
  X = "Bye"
  print X
f()
g()
```

**Program 2**

```
def f():
  S = "Hi"
  print S
def g():
  S = "Bye"
  print S
f()
g()
```

The output for both programs is

```
Hi
Bye
```

In each case the variable—whether X or S—is **local** to the function where it is defined. It is created each time the function is called and destroyed as soon as the call is complete. The local variable is *bound* by the function that defines it, and its *scope* is restricted to that function. Outside of the function definition the variable name can be used for any other purpose. That is why the functions f and g are allowed to use the same name for the variable in their definitions and why f and g define the same functions in both programs.

## *Implicit Quantification*

Consider the statement

If a number is an integer, then it is a rational number.

As shown earlier, this statement is equivalent to a universal statement. However, it does not contain the telltale word *all* or *every* or *any* or *each*. The only clue to indicate its universal quantification comes from the presence of the indefinite article *a*. This is an example of *implicit* universal quantification.

Existential quantification can also be implicit. For instance, the statement "The number 24 can be written as a sum of two even integers" can be expressed formally as "$\exists$ even integers $m$ and $n$ such that $24 = m + n$."

Mathematical writing contains many examples of implicitly quantified statements. Some occur, as in the first example above, through the presence of the word *a* or *an*. Others occur in cases where the general context of a sentence supplies part of its meaning. For example, in an algebra course in which the letter $x$ is always used to indicate a real number, the predicate

$$\text{If } x > 2 \text{ then } x^2 > 4$$

is interpreted to mean the same as the statement

$$\text{For every real number } x, \text{ if } x > 2 \text{ then } x^2 > 4.$$

Mathematicians often use a double arrow to indicate implicit quantification symbolically. For instance, they might express the above statement as

$$x > 2 \quad \Rightarrow \quad x^2 > 4.$$

### Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of $x$ is $D$.

- The notation $\boldsymbol{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x$, $P(x) \to Q(x)$.
- The notation $\boldsymbol{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x$, $P(x) \leftrightarrow Q(x)$.

**Example 3.1.12**

### Using $\Rightarrow$ and $\Leftrightarrow$

Let

$Q(n)$ be "$n$ is a factor of 8,"
$R(n)$ be "$n$ is a factor of 4,"
$S(n)$ be "$n < 5$ and $n \neq 3$,"

and suppose the domain of $n$ is $\mathbf{Z}^+$, the set of positive integers. Use the $\Rightarrow$ and $\Leftrightarrow$ symbols to indicate true relationships among $Q(n)$, $R(n)$, and $S(n)$.

#### Solution

1. As noted in Example 3.1.2, the truth set of $Q(n)$ is $\{1, 2, 4, 8\}$ when the domain of $n$ is $\mathbf{Z}^+$. By similar reasoning the truth set of $R(n)$ is $\{1, 2, 4\}$. Thus it is true that every element in the truth set of $R(n)$ is in the truth set of $Q(n)$, or, equivalently, $\forall n$ in $\mathbf{Z}^+$, $R(n) \to Q(n)$. So $R(n) \Rightarrow Q(n)$, or, equivalently
$$n \text{ is a factor of } 4 \quad \Rightarrow \quad n \text{ is a factor of } 8.$$
2. The truth set of $S(n)$ is $\{1, 2, 4\}$, which is identical to the truth set of $R(n)$, or, equivalently, $\forall n$ in $\mathbf{Z}^+$, $R(n) \leftrightarrow S(n)$. So $R(n) \Leftrightarrow S(n)$, or, equivalently,
$$n \text{ is a factor of } 4 \quad \Leftrightarrow \quad n < 5 \text{ and } n \neq 3.$$
   Moreover, since every element in the truth set of $S(n)$ is in the truth set of $Q(n)$, or, equivalently, $\forall n$ in $\mathbf{Z}^+$, $S(n) \to Q(n)$, then $S(n) \Rightarrow Q(n)$, or, equivalently,
$$n < 5 \text{ and } n \neq 3 \quad \Rightarrow \quad n \text{ is a factor of } 8.$$
■

Some questions of quantification can be quite subtle. For instance, a mathematics text might contain the following:

a. $(x+1)^2 = x^2 + 2x + 1$.
b. Solve $3x - 4 = 5$.

Although neither (a) nor (b) contains explicit quantification, the reader is supposed to understand that the $x$ in (a) is universally quantified, whereas the $x$ in (b) is existentially quantified. When the quantification is made explicit, (a) and (b) become

a. $\forall$ real number $x$, $(x+1)^2 = x^2 + 2x + 1$.
b. Show (by finding a value) that $\exists$ a real number $x$ such that $3x - 4 = 5$.

The quantification of a statement—whether universal or existential—crucially determines both how the statement can be applied and what method must be used to establish its truth. Thus it is important to be alert to the presence of hidden quantifiers when you read mathematics so that you will interpret statements in a logically correct way.

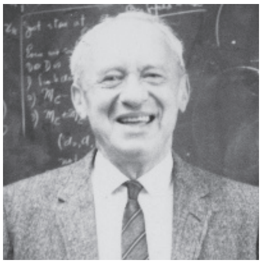
## *Tarski's World*

Tarski's World is a computer program developed by information scientists Jon Barwise and John Etchemendy to help teach the principles of logic. It is described in their book

*The Language of First-Order Logic*, which is accompanied by a CD containing the program Tarski's World, named after the great logician Alfred Tarski.

**Example 3.1.13**

### Investigating Tarski's World



Briscoe Center for American History

*Alfred Tarski (1902–1983)*

The program for Tarski's World provides pictures of blocks of various sizes, shapes, and colors, which are located on a grid. Shown in Figure 3.1.1 is a picture of an arrangement of objects in a two-dimensional Tarski world. The configuration can be described using logical operators and—for the two-dimensional version—notation such as Triangle($x$), meaning "$x$ is a triangle," Blue($y$), meaning "$y$ is blue," and RightOf($x$, $y$), meaning "$x$ is to the right of $y$ (but possibly in a different row)." Individual objects can be given names such as $a$, $b$, or $c$.

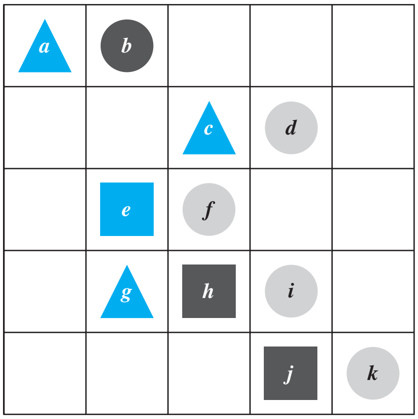


FIGURE 3.1.1

Determine the truth or falsity of each of the following statements. The domain for all variables is the set of objects in the Tarski world shown in Figure 3.1.1.

a. $\forall t$, Triangle($t$) $\longrightarrow$ Blue($t$).

b. $\forall x$, Blue($x$) $\longrightarrow$ Triangle($x$).

c. $\exists y$ such that Square($y$) $\wedge$ RightOf($d$, $y$).

d. $\exists z$ such that Square($z$) $\wedge$ Gray($z$).

#### Solution

a. This statement is true: Every triangle is blue.

b. This statement is false. As a counterexample, note that $e$ is blue and it is not a triangle.

c. This statement is true because $e$ and $h$ are both square and $d$ is to their right.

d. This statement is false: All the squares are either blue or black. ■

## TEST YOURSELF

Answers to Test Yourself questions are located at the end of each section.

1. If $P(x)$ is a predicate with domain $D$, the truth set of $P(x)$ is denoted ________. We read these symbols out loud as________.
2. Some ways to express the symbol $\forall$ in words are ________.
3. Some ways to express the symbol $\exists$ in words are ________.
4. A statement of the form $\forall x \in D$, $Q(x)$ is true if, and only if, $Q(x)$ is ________ for ________.
5. A statement of the form $\exists x \in D$ such that $Q(x)$ is true if, and only if, $Q(x)$ is ________ for ________.

## EXERCISE SET 3.1*

**1.** A menagerie consists of seven brown dogs, two black dogs, six gray cats, ten black cats, five blue birds, six yellow birds, and one black bird. Determine which of the following statements are true and which are false.
- **a.** There is an animal in the menagerie that is red.
- **b.** Every animal in the menagerie is a bird or a mammal.
- **c.** Every animal in the menagerie is brown or gray or black.
- **d.** There is an animal in the menagerie that is neither a cat nor a dog.
- **e.** No animal in the menagerie is blue.
- **f.** There are in the menagerie a dog, a cat, and a bird that all have the same color.

**2.** Indicate which of the following statements are true and which are false. Justify your answers as best as you can.
- **a.** Every integer is a real number.
- **b.** 0 is a positive real number.
- **c.** For every real number $r$, $-r$ is a negative real number.
- **d.** Every real number is an integer.

**3.** Let $R(m, n)$ be the predicate "If $m$ is a factor of $n^2$ then $m$ is a factor of $n$," with domain for both $m$ and $n$ being **Z** the set of integers.
- **a.** Explain why $R(m, n)$ is false if $m = 25$ and $n = 10$.
- **b.** Give values different from those in part (a) for which $R(m, n)$ is false.
- **c.** Explain why $R(m, n)$ is true if $m = 5$ and $n = 10$.
- **d.** Give values different from those in part (c) for which $R(m, n)$ is true.

**4.** Let $Q(x, y)$ be the predicate "If $x < y$ then $x^2 < y^2$" with domain for both $x$ and $y$ being **R** the set of real numbers.
- **a.** Explain why $Q(x, y)$ is false if $x = -2$ and $y = 1$.
- **b.** Give values different from those in part (a) for which $Q(x, y)$ is false.
- **c.** Explain why $Q(x, y)$ is true if $x = 3$ and $y = 8$.
- **d.** Give values different from those in part (c) for which $Q(x, y)$ is true.

**5.** Find the truth set of each predicate.
- **a.** Predicate: $6/d$ is an integer, domain: $\mathbf{Z}$
- **b.** Predicate: $6/d$ is an integer, domain: $\mathbf{Z}^+$
- **c.** Predicate: $1 \le x^2 \le 4$, domain: $\mathbf{R}$
- **d.** Predicate: $1 \le x^2 \le 4$, domain: $\mathbf{Z}$

**6.** Let $B(x)$ be "$-10 < x < 10$." Find the truth set of $B(x)$ for each of the following domains.
**a.** $\mathbf{Z}$ **b.** $\mathbf{Z}^+$ **c.** The set of all even integers

**7.** Let $S$ be the set of all strings of length 3 consisting of $a$'s, $b$'s, and $c$'s. List all the strings in $S$ that satisfy the following conditions:
1. Every string in $S$ begins with $b$.
2. No string in $S$ has more than one $c$.

**8.** Let $T$ be the set of all strings of length 3 consisting of 0's and 1's. List all the strings in $T$ that satisfy the following conditions:
1. For every string $s$ in $T$, the second character of $s$ is 1 or the first two characters of $s$ are the same.
2. No string in $T$ has all three characters the same.

**Find counterexamples to show that the statements in 9–12 are false.**

**9.** $\forall x \in \mathbf{R}, x \ge 1/x$.

**10.** $\forall a \in \mathbf{Z}, (a-1)/a$ is not an integer.

**11.** $\forall$ positive integers $m$ and $n$, $m \cdot n \ge m + n$.

**12.** $\forall$ real numbers $x$ and $y$, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

**13.** Consider the following statement:

$\forall$ basketball player $x$, $x$ is tall.

Which of the following are equivalent ways of expressing this statement?
- **a.** Every basketball player is tall.
- **b.** Among all the basketball players, some are tall.
- **c.** Some of all the tall people are basketball players.
- **d.** Anyone who is tall is a basketball player.
- **e.** All people who are basketball players are tall.
- **f.** Anyone who is a basketball player is a tall person.

---

*For exercises with blue numbers or letters, solutions are given in Appendix B. The symbol ***H*** indicates that only a hint or a partial solution is given. The symbol * signals that an exercise is more challenging than usual.

**14.** Consider the following statement:

$$\exists x \in \mathbf{R} \text{ such that } x^2 = 2.$$

Which of the following are equivalent ways of expressing this statement?

**a.** The square of each real number is 2.
**b.** Some real numbers have square 2.
**c.** The number $x$ has square 2, for some real number $x$.
**d.** If $x$ is a real number, then $x^2 = 2$.
**e.** Some real number has square 2.
**f.** There is at least one real number whose square is 2.

***H*** **15.** Rewrite the following statements informally in at least two different ways without using variables or quantifiers.

**a.** $\forall$ rectangle $x$, $x$ is a quadrilateral.
**b.** $\exists$ a set $A$ such that $A$ has 16 subsets.

**16.** Rewrite each of the following statements in the form "$\forall$ _____ $x$, _____."

**a.** All dinosaurs are extinct.
**b.** Every real number is positive, negative, or zero.
**c.** No irrational numbers are integers.
**d.** No logicians are lazy.
**e.** The number 2,147,581,953 is not equal to the square of any integer.
**f.** The number $-1$ is not equal to the square of any real number.

**17.** Rewrite each of the following in the form "$\exists$ _____ $x$ such that _____."

**a.** Some exercises have answers.
**b.** Some real numbers are rational.

**18.** Let $D$ be the set of all students at your school, and let $M(s)$ be "$s$ is a math major," let $C(s)$ be "$s$ is a computer science student," and let $E(s)$ be "$s$ is an engineering student." Express each of the following statements using quantifiers, variables, and the predicates $M(s)$, $C(s)$, and $E(s)$.

**a.** There is an engineering student who is a math major.
**b.** Every computer science student is an engineering student.
**c.** No computer science students are engineering students.
**d.** Some computer science students are also math majors.
**e.** Some computer science students are engineering students and some are not.

**19.** Consider the following statement:

$$\forall \text{ integer } n, \text{ if } n^2 \text{ is even then } n \text{ is even.}$$

Which of the following are equivalent ways of expressing this statement?

**a.** All integers have even squares and are even.
**b.** Given any integer whose square is even, that integer is itself even.
**c.** For all integers, there are some whose square is even.
**d.** Any integer with an even square is even.
**e.** If the square of an integer is even, then that integer is even.
**f.** All even integers have even squares.

***H*** **20.** Rewrite the following statement informally in at least two different ways without using variables or the symbol $\forall$ or the words "for all."

$\forall$ real numbers $x$, if $x$ is positive
then the square root of $x$ is positive.

**21.** Rewrite the following statements so that the quantifier trails the rest of the sentence.

**a.** For any graph $G$, the total degree of $G$ is even.
**b.** For any isosceles triangle $T$, the base angles of $T$ are equal.
**c.** There exists a prime number $p$ such that $p$ is even.
**d.** There exists a continuous function $f$ such that $f$ is not differentiable.

**22.** Rewrite each of the following statements in the form "$\forall$ _____ $x$, if _____ then _____."

**a.** All Java programs have at least 5 lines.
**b.** Any valid argument with true premises has a true conclusion.

**23.** Rewrite each of the following statements in the two forms "$\forall x$, if _____ then _____" and "$\forall x$, _____" (without an if-then).

**a.** All equilateral triangles are isosceles.
**b.** Every computer science student needs to take data structures.

**24.** Rewrite the following statements in the two forms "$\exists$ _____ $x$ such that _____" and "$\exists x$ such that _____ and _____."

**a.** Some hatters are mad.
**b.** Some questions are easy.

**25.** The statement "The square of any rational number is rational" can be rewritten formally as "For all rational numbers $x$, $x^2$ is rational" or as "For all $x$, if $x$ is rational then $x^2$ is rational." Rewrite each of the following statements in the two forms "$\forall$ ______ $x$, ______" and "$\forall x$, if ______, then ______" or in the two forms "$\forall$ ______ $x$ and $y$, ______" and "$\forall x$ and $y$, if ______, then ______."

**a.** The reciprocal of any nonzero fraction is a fraction.

**b.** The derivative of any polynomial function is a polynomial function.

**c.** The sum of the angles of any triangle is 180°.

**d.** The negative of any irrational number is irrational.

**e.** The sum of any two even integers is even.

**f.** The product of any two fractions is a fraction.

**26.** Consider the statement "All integers are rational numbers but some rational numbers are not integers."

**a.** Write this statement in the form "$\forall x$, if ______ then ______, but $\exists$ ______ $x$ such that ______."

**b.** Let Ratl($x$) be "$x$ is a rational number" and Int($x$) be "$x$ is an integer." Write the given statement formally using only the symbols Ratl($x$), Int($x$), $\forall$, $\exists$, $\wedge$, $\vee$, $\sim$, and $\rightarrow$.

**27.** Refer to the picture of Tarski's world given in Example 3.1.13. Let Above($x$, $y$) mean that $x$ is above $y$ (but possibly in a different column). Determine the truth or falsity of each of the following statements. Give reasons for your answers.

**a.** $\forall u$, Circle$(u) \rightarrow$ Gray$(u)$.

**b.** $\forall u$, Gray$(u) \rightarrow$ Circle$(u)$.

**c.** $\exists y$ such that Square$(y) \wedge$ Above$(y, d)$.

**d.** $\exists z$ such that Triangle$(z) \wedge$ Above$(f, z)$.

**In 28–30, rewrite each statement without using quantifiers or variables. Indicate which are true and which are false, and justify your answers as best as you can.**

**28.** Let the domain of $x$ be the set $D$ of objects discussed in mathematics courses, and let Real($x$) be "$x$ is a real number," Pos($x$) be "$x$ is a positive real number," Neg($x$) be "$x$ is a negative real number," and Int($x$) be "$x$ is an integer."

**a.** Pos(0)

**b.** $\forall x$, Real$(x) \wedge$ Neg$(x) \rightarrow$ Pos$(-x)$

**c.** $\forall x$, Int$(x) \rightarrow$ Real$(x)$

**d.** $\exists x$ such that Real$(x) \wedge \sim$Int$(x)$

**29.** Let the domain of $x$ be the set of geometric figures in the plane, and let Square($x$) be "$x$ is a square" and Rect($x$) be "$x$ is a rectangle."

**a.** $\exists x$ such that Rect$(x) \wedge$ Square$(x)$

**b.** $\exists x$ such that Rect$(x) \wedge \sim$Square$(x)$

**c.** $\forall x$, Square$(x) \rightarrow$ Rect$(x)$

**30.** Let the domain of $x$ be $\mathbf{Z}$, the set of integers, and let Odd($x$) be "$x$ is odd," Prime($x$) be "$x$ is prime," and Square($x$) be "$x$ is a perfect square." (An integer $n$ is said to be a **perfect square** if, and only if, it equals the square of some integer. For example, 25 is a perfect square because $25 = 5^2$.)

**a.** $\exists x$ such that Prime$(x) \wedge \sim$Odd$(x)$

**b.** $\forall x$, Prime$(x) \rightarrow \sim$Square$(x)$

**c.** $\exists x$ such that Odd$(x) \wedge$ Square$(x)$

*H* **31.** In any mathematics or computer science text other than this book, find an example of a statement that is universal but is implicitly quantified. Copy the statement as it appears and rewrite it making the quantification explicit. Give a complete citation for your example, including title, author, publisher, year, and page number.

**32.** Let $\mathbf{R}$ be the domain of the predicate variable $x$. Which of the following are true and which are false? Give counter examples for the statements that are false.

**a.** $x > 2 \Rightarrow x > 1$

**b.** $x > 2 \Rightarrow x^2 > 4$

**c.** $x^2 > 4 \Rightarrow x > 2$

**d.** $x^2 > 4 \Leftrightarrow |x| > 2$

**33.** Let $\mathbf{R}$ be the domain of the predicate variables $a$, $b$, $c$, and $d$. Which of the following are true and which are false? Give counterexamples for the statements that are false.

**a.** $a > 0$ and $b > 0 \Rightarrow ab > 0$

**b.** $a < 0$ and $b < 0 \Rightarrow ab < 0$

**c.** $ab = 0 \Rightarrow a = 0$ or $b = 0$

**d.** $a < b$ and $c < d \Rightarrow ac < bd$

## ANSWERS FOR TEST YOURSELF

**1.** $\{x \in D \mid P(x)\}$; the set of all $x$ in $D$ such that $P(x)$ **2.** *Possible answers:* for every, for any, for each, for arbitrary, given any, for all **3.** *Possible answers:* there exists, there exist, there exists at least one, for some, for at least one, we can find a **4.** true; every $x$ in $D$ (*Some alternative answers:* all $x$ in $D$; each individual $x$ in $D$) **5.** true; at least one $x$ in $D$ (*Alternative answer:* some $x$ in $D$)

## 3.2 Predicates and Quantified Statements II

TOUCHSTONE: *Stand you both forth now: stroke your chins, and swear by your beards that I am a knave.*
CELIA: *By our beards—if we had them—thou art.*
TOUCHSTONE: *By my knavery—if I had it—then I were; but if you swear by that that is not, you are not forsworn.* —William Shakespeare, *As You Like It*

This section continues the discussion of predicates and quantified statements begun in Section 3.1. It contains the rules for negating quantified statements; an exploration of the relation among $\forall$, $\exists$, $\wedge$, and $\vee$; an introduction to the concept of vacuous truth of universal statements; examples of variants of universal conditional statements; and an extension of the meaning of *necessary*, *sufficient*, and *only if* to quantified statements.

### *Negations of Quantified Statements*

Consider the statement "All mathematicians wear glasses." Many people would say that its negation is "No mathematicians wear glasses," but if even one mathematician does not wear glasses, then the sweeping statement that *all* mathematicians wear glasses is false. So a correct negation is "There is at least one mathematician who does not wear glasses."

The general form of the negation of a universal statement follows immediately from the definitions of negation and of the truth values for universal and existential statements.

**Theorem 3.2.1 Negation of a Universal Statement**

The negation of a statement of the form

$$\forall x \text{ in } D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically,

$$\sim(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$$

Thus

> **The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").**

Note that when we speak of **logical equivalence for quantified statements**, we mean that the statements always have identical truth values no matter what predicates are substituted for the predicate symbols and no matter what sets are used for the domains of the predicate variables.

Now consider the statement "Some snowflakes are the same." What is its negation? For this statement to be false means that not a single snowflake is the same as any other. In other words, "No snowflakes are the same," or "All snowflakes are different."

The general form for the negation of an existential statement follows immediately from the definitions of negation and of the truth values for existential and universal statements.

**Theorem 3.2.2 Negation of an Existential Statement**

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Symbolically,

$$\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$$

Thus

**The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").**

**Example 3.2.1**

### Negating Quantified Statements

Write formal negations for the following statements:

a. $\forall$ primes $p$, $p$ is odd.

b. $\exists$ a triangle $T$ such that the sum of the angles of $T$ equals 200°.

#### Solution

a. By applying the rule for the negation of a $\forall$ statement, you can see that the answer is

$$\exists \text{ a prime } p \text{ such that } p \text{ is not odd.}$$

b. By applying the rule for the negation of a $\exists$ statement, you can see that the answer is

$$\forall \text{ triangles } T, \text{ the sum of the angles of } T \text{ does not equal } 200°.$$ ■

You need to exercise special care to avoid mistakes when writing negations of statements that are given informally. One way to avoid error is to rewrite the statement formally and take the negation using the formal rule.

**Example 3.2.2**

### More Negations

Rewrite the following statements formally. Then write formal and informal negations.

a. No politicians are honest.

b. The number 1,357 is not divisible by any integer between 1 and 37.

#### Solution

a. *Formal version:* $\forall$ politicians $x$, $x$ is not honest.
*Formal negation:* $\exists$ a politician $x$ such that $x$ is honest.
*Informal negation:* Some politicians are honest.

b. This statement has a trailing quantifier. Written formally it becomes:

$$\forall \text{ integer } n \text{ between 1 and 37, 1,357 is not divisible by } n.$$

**Note** Which is true: the statement in part (b) or its negation? Is 1,357 divisible by some integer between 1 and 37? Or is 1,357 not divisible by any integer between 1 and 37?

Its negation is therefore

$$\exists \text{ an integer } n \text{ between 1 and 37 such that 1,357 is divisible by } n.$$

An informal version of the negation is

The number 1,357 is divisible by some integer between 1 and 37. ■

Another important way to avoid error when taking negations of statements, whether stated formally or informally, is to ask yourself, "What *exactly* would it mean for the given statement to be false? What statement, if true, would be equivalent to saying that the given statement is false?"

### Example 3.2.3 Still More Negations

Write informal negations for the following statements:

a. All computer programs are finite.

b. Some computer hackers are over 40.

### Solution

a. What exactly would it mean for this statement to be false? The statement asserts that all computer programs satisfy a certain property. So for it to be false, there would have to be at least one computer program that does not satisfy the property. Thus the answer is

There is a computer program that is not finite.

*Or:* Some computer programs are infinite.

b. This statement is equivalent to saying that there is at least one computer hacker with a certain property. So for it to be false, not a single computer hacker can have that property. Thus the negation is

No computer hackers are over 40.

*Or:* All computer hackers are 40 or under. ■

**Caution!** Just inserting the word *not* to negate a quantified statement can result in a statement that is ambiguous.

Informal negations of many universal statements can be constructed simply by inserting the word *not* or the words *do not* at an appropriate place. However, the resulting statements may be ambiguous. For example, a possible negation of "All mathematicians wear glasses" is "All mathematicians do not wear glasses." The problem is that this sentence has two meanings. With the proper verbal stress on the word *not*, it could be interpreted as the logical negation. (What! You say that all mathematicians wear glasses? Nonsense! All mathematicians do *not* wear glasses.) On the other hand, stated in a flat tone of voice (try it!), it would mean that all mathematicians are nonwearers of glasses; that is, not a single mathematician wears glasses. This is a much stronger statement than the logical negation: It implies the negation but is not equivalent to it.

## *Negations of Universal Conditional Statements*

Negations of universal conditional statements are of special importance in mathematics. The form of such negations can be derived from facts that have already been established.

By definition of the negation of a *for all* statement,

$$\sim(\forall x, P(x) \to Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \to Q(x)). \qquad 3.2.1$$

But the negation of an if-then statement is logically equivalent to an *and* statement. More precisely,

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x). \qquad 3.2.2$$

Substituting (3.2.2) into (3.2.1) gives

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Written somewhat less symbolically, this becomes

### Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

**Example 3.2.4**

### Negating Universal Conditional Statements

Write a formal negation for statement (a) and an informal negation for statement (b).

a. ∀ person $p$, if $p$ is blond then $p$ has blue eyes.

b. If a computer program has more than 100,000 lines, then it contains a bug.

**Solution**

a. ∃ a person $p$ such that $p$ is blond and $p$ does not have blue eyes.

b. There is at least one computer program that has more than 100,000 lines and does not contain a bug. ■

## *The Relation among ∀, ∃, ∧, and ∨*

The negation of a *for all* statement is a *there exists* statement, and the negation of a *there exists* statement is a *for all* statement. These facts are analogous to De Morgan's laws, which state that the negation of an *and* statement is an *or* statement and that the negation of an *or* statement is an *and* statement. This similarity is not accidental. In a sense, universal statements are generalizations of *and* statements, and existential ments are generalizations of *or* statements.

If $Q(x)$ is a predicate and the domain $D$ of $x$ is the set $\{x_1, x_2, \ldots, x_n\}$, then the statements

$$\forall x \in D, Q(x) \quad \text{and} \quad Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n)$$

are logically equivalent. For example, let $Q(x)$ be "$x \cdot x = x$" and suppose $D = \{0, 1\}$. Then

$$\forall x \in D, Q(x)$$

can be rewritten as

$$\forall \text{ binary digits } x, x \cdot x = x.$$

This is equivalent to

$$0 \cdot 0 = 0 \quad \text{and} \quad 1 \cdot 1 = 1,$$

which can be rewritten in symbols as

$$Q(0) \wedge Q(1).$$

Similarly, if $Q(x)$ is a predicate and $D = \{x_1, x_2, \ldots, x_n\}$, then the statements

$$\exists x \in D \text{ such that } Q(x) \quad \text{and} \quad Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n)$$

are logically equivalent. For example, let $Q(x)$ be "$x + x = x$" and suppose $D = \{0, 1\}$. Then

$$\exists x \in D \text{ such that } Q(x)$$

can be rewritten as

$$\exists \text{ a binary digit } x \text{ such that } x + x = x.$$

This is equivalent to

$$0 + 0 = 0 \quad \text{or} \quad 1 + 1 = 1,$$

which can be rewritten in symbols as

$$Q(0) \vee Q(1).$$

## Vacuous Truth of Universal Statements

Suppose a bowl sits on a table and next to the bowl is a pile of five blue and five gray balls, any of which may be placed in the bowl. If three blue balls and one gray ball are placed in the bowl, as shown in Figure 3.2.1(a), the statement "All the balls in the bowl are blue" would be false (since one of the balls in the bowl is gray).

Now suppose that no balls at all are placed in the bowl, as shown in Figure 3.2.1(b). Consider the statement

All the balls in the bowl are blue.

Is this statement true or false? The statement is false if, and only if, its negation is true. And its negation is

There exists a ball in the bowl that is not blue.

But the only way this negation can be true is for there actually to be a nonblue ball in the bowl. And there is not! Hence the negation is false, and so the statement is true "by default."

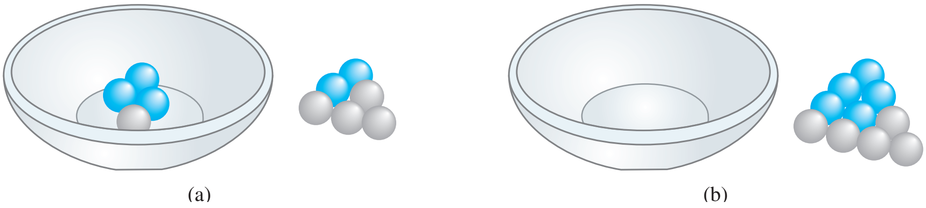

(a) (b)

**FIGURE 3.2.1**

**Note** In ordinary language the words *in general* mean that something is usually, but not always the case. (In general, I take the bus, but today I walked.) In mathematics the words *in general* mean that something is always true.

In general, a statement of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

is called **vacuously true** or **true by default** if, and only if, $P(x)$ is false for every $x$ in $D$.

In mathematics, the words *in general* signal that what is to follow is a generalization of some aspect of the example that always holds true.

## Variants of Universal Conditional Statements

Recall from Section 2.2 that a conditional statement has a contrapositive, a converse, and an inverse. The definitions of these terms can be extended to universal conditional statements.

**Definition**

Consider a statement of the form $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

### Example 3.2.5 Contrapositive, Converse, and Inverse of a Universal Conditional Statement

Write a formal and an informal contrapositive, converse, and inverse for the following statement:

If a real number is greater than 2, then its square is greater than 4.

**Solution** The formal version of this statement is $\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.

*Contrapositive:* $\forall x \in \mathbf{R}$, if $x^2 \leq 4$ then $x \leq 2$.
*Or:* If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

*Converse:* $\forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$.
*Or:* If the square of a real number is greater than 4, then the number is greater than 2.

*Inverse:* $\forall x \in \mathbf{R}$, if $x \leq 2$ then $x^2 \leq 4$.
*Or:* If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

Note that in solving this example, we have used the equivalence of "$x \not> a$" and "$x \leq a$" for all real numbers $x$ and $a$. (See page 47.) ■

In Section 2.2 we showed that a conditional statement is logically equivalent to its contrapositive and that it is not logically equivalent to either its converse or its inverse. The following discussion shows that these facts generalize to the case of universal conditional statements and their contrapositives, converses, and inverses.

Let $P(x)$ and $Q(x)$ be any predicates, let $D$ be the domain of $x$, and consider the statement

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

and its contrapositive

$$\forall x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x).$$

Any particular $x$ in $D$ that makes "if $P(x)$ then $Q(x)$" true also makes "if $\sim Q(x)$ then $\sim P(x)$" true (by the logical equivalence between $p \to q$ and $\sim q \to \sim p$). It follows that the sentence "If $P(x)$ then $Q(x)$" is true for all $x$ in $D$ if, and only if, the sentence "If $\sim Q(x)$ then $\sim P(x)$" is true for all $x$ in $D$.

Thus we write the following and say that a universal conditional statement is logically equivalent to its contrapositive:

$$\boxed{\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \text{ if } \sim Q(x) \text{ then } \sim P(x)}$$

In Example 3.2.5 we noted that the statement

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4$$

has the converse $\quad \forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$.

Observe that the statement is true whereas its converse is false (since, for instance, $(-3)^2 = 9 > 4$ but $-3 \not> 2$). This shows that a universal conditional statement may have a different truth value from its converse. Hence a universal conditional statement is not logically equivalent to its converse. This is written in symbols as follows:

$$\boxed{\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \not\equiv \forall x \in D, \text{ if } Q(x) \text{ then } P(x).}$$

In exercise 35 at the end of this section, you are asked to provide an example to show that a universal conditional statement is not logically equivalent to its inverse.

$$\boxed{\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \not\equiv \forall x \in D, \text{ if } \sim P(x) \text{ then } \sim Q(x).}$$

## *Necessary and Sufficient Conditions, Only If*

The definitions of *necessary*, *sufficient*, and *only if* can also be extended to apply to universal conditional statements.

### Definition

- "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."
- "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."
- "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

**Example 3.2.6**

### Necessary and Sufficient Conditions

Rewrite each of the following as a universal conditional statement, quantified either explicitly or implicitly. Do not use the word *necessary* or *sufficient*.

a. Squareness is a sufficient condition for rectangularity.

b. Being at least 35 years old is a necessary condition for being president of the United States.

### Solution

a. A formal version of the statement is

$\forall x$, if $x$ is a square, then $x$ is a rectangle.

Or, with implicit universal quantification:

If a figure is a square, then it is a rectangle.

b. Using formal language, you could write the answer as

$\forall$ person $x$, if $x$ is younger than 35, then $x$
cannot be president of the United States.

Or, by the equivalence between a statement and its contrapositive:

$\forall$ person $x$, if $x$ is president of the United States,
then $x$ is at least 35 years old. ■

### **Example 3.2.7** Only If

Rewrite the following as a universal conditional statement:

A product of two numbers is 0 only if one of the numbers is 0.

**Solution** Using informal language, you could write the answer as

If it is not the case that one of two numbers is 0,
then the product of the numbers is not 0.

In other words,

If neither of two numbers is 0, then the product of the numbers is not 0.

Or, by the equivalence between a statement and its contrapositive:

If a product of two numbers is 0, then one of the numbers is 0. ■

## TEST YOURSELF

1. A negation for "All $R$ have property $S$" is "There is _____ $R$ that _____."
2. A negation for "Some $R$ have property $S$" is "_____."
3. A negation for "For every $x$, if $x$ has property $P$ then $x$ has property $Q$" is "_____."
4. The converse of "For every $x$, if $x$ has property $P$ then $x$ has property $Q$" is "_____."
5. The contrapositive of "For every $x$, if $x$ has property $P$ then $x$ has property $Q$" is "_____."
6. The inverse of "For every $x$, if $x$ has property $P$ then $x$ has property $Q$" is "_____."

## EXERCISE SET 3.2

1. Which of the following is a negation for "All discrete mathematics students are athletic"? More than one answer may be correct.
   - **a.** There is a discrete mathematics student who is nonathletic.
   - **b.** All discrete mathematics students are nonathletic.
   - **c.** There is an athletic person who is not a discrete mathematics student.
   - **d.** No discrete mathematics students are athletic.
   - **e.** Some discrete mathematics students are nonathletic.
   - **f.** No athletic people are discrete mathematics students.
2. Which of the following is a negation for "All dogs are loyal"? More than one answer may be correct.
   - **a.** All dogs are disloyal.
   - **b.** No dogs are loyal.
   - **c.** Some dogs are disloyal.
   - **d.** Some dogs are loyal.
   - **e.** There is a disloyal animal that is not a dog.
   - **f.** There is a dog that is disloyal.
   - **g.** No animals that are not dogs are loyal.
   - **h.** Some animals that are not dogs are loyal.
3. Write a formal negation for each of the following statements.
   - **a.** $\forall$ string $s$, $s$ has at least one character.
   - **b.** $\forall$ computer $c$, $c$ has a CPU.
   - **c.** $\exists$ a movie $m$ such that $m$ is over 6 hours long.
   - **d.** $\exists$ a band $b$ such that $b$ has won at least 10 Grammy awards.
4. Write an informal negation for each of the following statements. Be careful to avoid negations that are ambiguous.
   - **a.** All dogs are friendly.
   - **b.** All graphs are connected.
   - **c.** Some suspicions were substantiated.
   - **d.** Some estimates are accurate.
5. Write a negation for each of the following statements.
   - **a.** Every valid argument has a true conclusion.
   - **b.** All real numbers are positive, negative, or zero.

**Write a negation for each statement in 6 and 7.**

**6.** **a.** Sets $A$ and $B$ do not have any points in common.
**b.** Towns $P$ and $Q$ are not connected by any road on the map.

**7.** **a.** This vertex is not connected to any other vertex in the graph.
**b.** This number is not related to any even number.

**8.** Consider the statement "There are no simple solutions to life's problems." Write an informal negation for the statement, and then write the statement formally using quantifiers and variables.

**Write a negation for each statement in 9 and 10.**

**9.** $\forall$ real number $x$, if $x > 3$ then $x^2 > 9$.

**10.** $\forall$ computer program $P$, if $P$ compiles without error messages, then $P$ is correct.

**In each of 11–14 determine whether the proposed negation is correct. If it is not, write a correct negation.**

**11.** *Statement:* The sum of any two irrational numbers is irrational.
*Proposed negation:* The sum of any two irrational numbers is rational.

**12.** *Statement:* The product of any irrational number and any rational number is irrational.
*Proposed negation:* The product of any irrational number and any rational number is rational.

**13.** *Statement:* For every integer $n$, if $n^2$ is even then $n$ is even.
*Proposed negation:* For every integer $n$, if $n^2$ is even then $n$ is not even.

**14.** *Statement:* For all real numbers $x_1$ and $x_2$, if $x_1^2 = x_2^2$ then $x_1 = x_2$.
*Proposed negation:* For all real numbers $x_1$ and $x_2$, if $x_1^2 = x_2^2$ then $x_1 \neq x_2$.

**15.** Let $D = \{-48, -14, -8, 0, 1, 3, 16, 23, 26, 32, 36\}$. Determine which of the following statements are true and which are false. Provide counterexamples for the statements that are false.
**a.** $\forall x \in D$, if $x$ is odd then $x > 0$.
**b.** $\forall x \in D$, if $x$ is less than 0 then $x$ is even.
**c.** $\forall x \in D$, if $x$ is even then $x \leq 0$.
**d.** $\forall x \in D$, if the ones digit of $x$ is 2, then the tens digit is 3 or 4.
**e.** $\forall x \in D$, if the ones digit of $x$ is 6, then the tens digit is 1 or 2.

**In 16–23, write a negation for each statement.**

**16.** $\forall$ real number $x$, if $x^2 \geq 1$ then $x > 0$.

**17.** $\forall$ integer $d$, if $6/d$ is an integer then $d = 3$.

**18.** $\forall x \in \mathbf{R}$, if $x(x+1) > 0$ then $x > 0$ or $x < -1$.

**19.** $\forall n \in \mathbf{Z}$, if $n$ is prime then $n$ is odd or $n = 2$.

**20.** $\forall$ integers $a$, $b$, and $c$, if $a - b$ is even and $b - c$ is even, then $a - c$ is even.

**21.** $\forall$ integer $n$, if $n$ is divisible by 6, then $n$ is divisible by 2 and $n$ is divisible by 3.

**22.** If the square of an integer is odd, then the integer is odd.

**23.** If a function is differentiable then it is continuous.

**24.** Rewrite the statements in each pair in if-then form and indicate the logical relationship between them.
**a.** All the children in Tom's family are female.
All the females in Tom's family are children.
**b.** All the integers that are greater than 5 and end in 1, 3, 7, or 9 are prime.
All the integers that are greater than 5 and are prime end in 1, 3, 7, or 9.

**25.** Each of the following statements is true. In each case write the converse of the statement, and give a counterexample showing that the converse is false.
**a.** If $n$ is any prime number that is greater than 2, then $n + 1$ is even.
**b.** If $m$ is any odd integer, then $2m$ is even.
**c.** If two circles intersect in exactly two points, then they do not have a common center.

**In 26–33, for each statement in the referenced exercise write the contrapositive, converse, and inverse. Indicate as best as you can which of these statements are true and which are false. Give a counterexample for each that is false.**

**26.** Exercise 16
**27.** Exercise 17
**28.** Exercise 18
**29.** Exercise 19
**30.** Exercise 20
**31.** Exercise 21
**32.** Exercise 22
**33.** Exercise 23

**34.** Write the contrapositive for each of the following statements.
**a.** If $n$ is prime, then $n$ is not divisible by any prime number from 2 through $\sqrt{n}$. (Assume that $n$ is a fixed integer.)
**b.** If $A$ and $B$ do not have any elements in common, then they are disjoint. (Assume that $A$ and $B$ are fixed sets.)

**35.** Give an example to show that a universal conditional statement is not logically equivalent to its inverse.

***36.** If $P(x)$ is a predicate and the domain of $x$ is the set of all real numbers, let $R$ be "$\forall x \in \mathbf{Z}, P(x)$," let $S$ be "$\forall x \in \mathbf{Q}, P(x)$," and let $T$ be "$\forall x \in \mathbf{R}, P(x)$."

- **a.** Find a definition for $P(x)$ (but do not use "$x \in \mathbf{Z}$") so that $R$ is true and both $S$ and $T$ are false.
- **b.** Find a definition for $P(x)$ (but do not use "$x \in \mathbf{Q}$") so that both $R$ and $S$ are true and $T$ is false.

**37.** Consider the following sequence of digits: 0204. A person claims that all the 1's in the sequence are to the left of all the 0's in the sequence. Is this true? Justify your answer. (*Hint:* Write the claim formally and write a formal negation for it. Is the negation true or false?)

**38.** True or false? All occurrences of the letter *u* in *Discrete Mathematics* are lowercase. Justify your answer.

**Rewrite each statement of 39–44 in if-then form.**

**39.** Earning a grade of C− in this course is a sufficient condition for it to count toward graduation.

**40.** Being divisible by 8 is a sufficient condition for being divisible by 4.

**41.** Being on time each day is a necessary condition for keeping this job.

**42.** Passing a comprehensive exam is a necessary condition for obtaining a master's degree.

**43.** A number is prime only if it is greater than 1.

**44.** A polygon is square only if it has four sides.

**Use the facts that the negation of a $\forall$ statement is a $\exists$ statement and that the negation of an if-then statement is an *and* statement to rewrite each of the statements 45–48 without using the word *necessary* or *sufficient*.**

**45.** Being divisible by 8 is not a necessary condition for being divisible by 4.

**46.** Having a large income is not a necessary condition for a person to be happy.

**47.** Having a large income is not a sufficient condition for a person to be happy.

**48.** Being a polynomial is not a sufficient condition for a function to have a real root.

**49.** The computer scientists Richard Conway and David Gries once wrote:

> The absence of error messages during translation of a computer program is only a necessary and not a sufficient condition for reasonable [program] correctness.

Rewrite this statement without using the words *necessary* or *sufficient*.

**50.** A frequent-flyer club brochure states, "You may select among carriers only if they offer the same lowest fare." Assuming that "only if" has its formal, logical meaning, does this statement guarantee that if two carriers offer the same lowest fare, the customer will be free to choose between them? Explain.

## ANSWERS FOR TEST YOURSELF

**1.** some (*Alternative answers:* at least one; an); does not have property $S$. **2.** No $R$ have property $S$. **3.** There is an $x$ such that $x$ has property $P$ and $x$ does not have property $Q$. **4.** For every $x$, if $x$ has property $Q$ then $x$ has property $P$. **5.** For every $x$, if $x$ does not have property $Q$ then $x$ does not have property $P$. **6.** For every $x$, if $x$ does not have property $P$ then $x$ does not have property $Q$.

# 3.3 Statements with Multiple Quantifiers

*It is not enough to have a good mind. The main thing is to use it well.* —René Descartes

Imagine you are visiting a factory that manufactures computer microchips. The factory guide tells you,

"There is a person supervising every detail of the production process."

Note that this statement contains informal versions of both the existential quantifier *there is* and the universal quantifier *every*. Which of the following best describes its meaning?

- There is one single person who supervises all the details of the production process.
- For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details.

As it happens, either interpretation could be what the guide meant. (Reread the sentence to be sure you agree!) Taken by itself, his statement is genuinely ambiguous, although other things he may have said (the context for his statement) might have clarified it. In our ordinary lives, we deal with this kind of ambiguity all the time. Usually context helps resolve it, but sometimes we simply misunderstand each other.

In mathematics, formal logic, and computer science, by contrast, it is essential that we all interpret statements in exactly the same way. For instance, the initial stage of software development typically involves careful discussion between a programmer analyst and a client to turn vague descriptions of what the client wants into unambiguous program specifications that client and programmer can mutually agree on.

Because many important technical statements contain both $\exists$ and $\forall$, a convention has developed for interpreting them uniformly. ***When a statement contains more than one kind of quantifier, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur.*** For instance, consider a statement of the form

$$\forall x \text{ in set } D, \exists y \text{ in set } E \text{ such that } x \text{ and } y \text{ satisfy property } P(x, y).$$

To show that such a statement is true, you must be able to meet the following challenge:

**Note** The scope of $\forall x$ extends throughout the statement, whereas the scope of $\exists y$ starts in the middle. That is why the value of $y$ depends on the value of $x$.

- Imagine that someone is allowed to choose any element whatsoever from the set $D$, and imagine that the person gives you that element. Call it $x$.
- The challenge for you is to find an element $y$ in $E$ so that the person's $x$ and your $y$, taken together, satisfy property $P(x, y)$.

***Because you do not have to specify the y until after the other person has specified the x, you are allowed to find a different value of y for each different x you are given.***

## Example 3.3.1 Truth of a $\forall\exists$ Statement in a Tarski World

Consider the Tarski world shown in Figure 3.3.1.

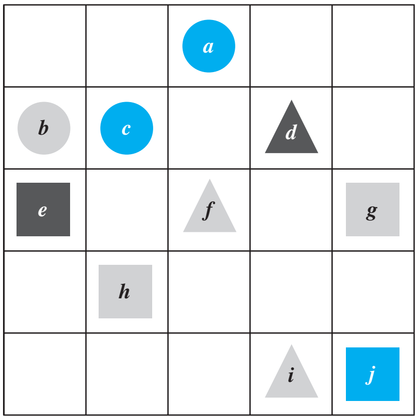


FIGURE 3.3.1

Show that the following statement is true in this world:

For every triangle $x$, there is a square $y$ such that $x$ and $y$ have the same color.

**Solution** The statement says that no matter which triangle someone gives you, you will be able to find a square of the same color. There are only three triangles, $d$, $f$, and $i$. The following table shows that for each of these triangles a square of the same color can be found.

| Given $x =$ | choose $y =$ | and check that $y$ is the same color as $x$. |
|---|---|---|
| $d$ | $e$ | yes ✓ |
| $f$ or $i$ | $h$ or $g$ | yes ✓ |

■

Now consider a statement containing both ∀ and ∃, where the ∃ comes before the ∀:

$$\exists x \text{ in set } D \text{ such that } \forall y \text{ in set } E,\ x \text{ and } y \text{ satisfy property } P(x, y).$$

To show that a statement of this form is true:

You must find one single element (call it $x$) in $D$ with the following property:

- After you have found your $x$, someone is allowed to choose any element whatsoever from $E$. The person challenges you by giving you that element. Call it $y$.
- Your job is to show that your $x$ together with the person's $y$ satisfy property $P(x, y)$.

Your $x$ has to work for *any* $y$ the person might give you; ***you are not allowed to change your x once you have specified it initially.***

**Note** The value of $x$ cannot be changed once it is specified because the scope of $\exists x$ extends throughout the entire statement.

**Example 3.3.2**

## Truth of a ∃∀ Statement in a Tarski World

Consider again the Tarski world in Figure 3.3.1. Show that the following statement is true: There is a triangle $x$ such that for every circle $y$, $x$ is to the right of $y$.

**Solution** The statement says that you can find a triangle that is to the right of all the circles. Actually, either $d$ or $i$ would work for all of the three circles, $a$, $b$, and $c$, as you can see in the following table.

| Choose $x =$ | Then: given $y =$ | check that $x$ is to the right of $y$. |
|---|---|---|
| $d$ or $i$ | $a$ | yes ✓ |
|  | $b$ | yes ✓ |
|  | $c$ | yes ✓ |

■

Here is a summary of the convention for interpreting statements with two different quantifiers:

### Interpreting Statements with Two Different Quantifiers

If you want to establish the truth of a statement of the form

$$\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)$$

your challenge is to allow someone else to pick whatever element $x$ in $D$ they wish and then you must find an element $y$ in $E$ that "works" for that particular $x$.

If you want to establish the truth of a statement of the form

$$\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)$$

your job is to find one particular $x$ in $D$ that will "work" no matter what $y$ in $E$ anyone might choose to challenge you with.

### Example 3.3.3 Interpreting Statements with More Than One Quantifier

A college cafeteria line has four stations: salads, main courses, desserts, and beverages. The salad station offers a choice of green salad or fruit salad; the main course station offers spaghetti or fish; the dessert station offers pie or cake; and the beverage station offers milk, soda, or coffee. Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

Uta: green salad, spaghetti, pie, milk
Tim: fruit salad, fish, pie, cake, milk, coffee
Yuen: spaghetti, fish, pie, soda

These choices are illustrated in Figure 3.3.2.

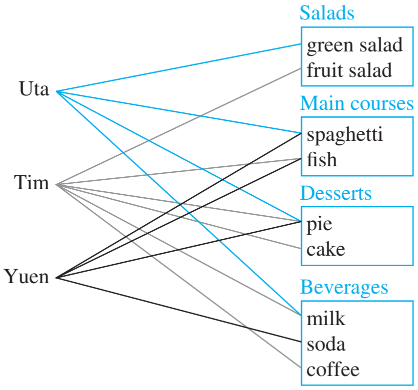


FIGURE 3.3.2

Write each of following statements informally and find its truth value.

a. $\exists$ an item $I$ such that $\forall$ student $S$, $S$ chose $I$.

b. $\exists$ a student $S$ such that $\forall$ item $I$, $S$ chose $I$.

c. $\exists$ a student $S$ such that $\forall$ station $Z$, $\exists$ an item $I$ in $Z$ such that $S$ chose $I$.

d. $\forall$ student $S$ and $\forall$ station $Z$, $\exists$ an item $I$ in $Z$ such that $S$ chose $I$.

#### Solution

a. There is an item that was chosen by every student. This is true; every student chose pie.

b. There is a student who chose every available item. This is false; no student chose all nine items.

c. There is a student who chose at least one item from every station. This is true; both Uta and Tim chose at least one item from every station.

d. Every student chose at least one item from every station. This is false; Yuen did not choose a salad. ■

## Translating from Informal to Formal Language

Most problems are stated in informal language, but solving them often requires translating them into more formal terms.

### Example 3.3.4 Translating Statements with Multiple Quantifiers from Informal to Formal Language

The **reciprocal** of a real number $a$ is a real number $b$ such that $ab = 1$. The following two statements are true. Rewrite them formally using quantifiers and variables.

a. Every nonzero real number has a reciprocal.

b. There is a real number with no reciprocal.

**Note** The number 0 has no reciprocal.

**Solution**

a. $\forall$ nonzero real number $u$, $\exists$ a real number $v$ such that $uv = 1$.

b. $\exists$ a real number $c$ such that $\forall$ real number $d$, $cd \neq 1$. ■

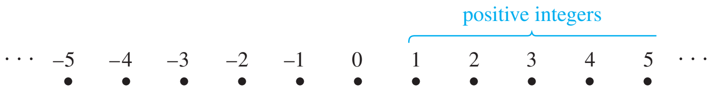
### Example 3.3.5 There Is a Smallest Positive Integer

Recall that every integer is a real number and that real numbers are of three types: positive, negative, and zero (zero being neither positive nor negative). Consider the statement "There is a smallest positive integer." Write this statement formally using both symbols $\exists$ and $\forall$.

**Solution** To say that there is a smallest positive integer means that there is a positive integer $m$ with the property that no matter what positive integer $n$ a person might pick, $m$ will be less than or equal to $n$:

$$\exists \text{ a positive integer } m \text{ such that } \forall \text{ positive integer } n,\ m \leq n.$$

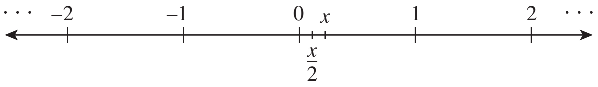
Note that this statement is true because 1 is a positive integer that is less than or equal to every positive integer.



■

### Example 3.3.6 There Is No Smallest Positive Real Number

Imagine the positive real numbers on the real number line. These numbers correspond to all the points to the right of 0. Observe that no matter how small a real number $x$ is, the number $x/2$ will be both positive and less than $x$.*



Thus the following statement is true: "There is no smallest positive real number." Write this statement formally using both symbols $\forall$ and $\exists$.

**Solution** $\forall$ positive real number $x$, $\exists$ a positive real number $y$ such that $y < x$. ■

### Example 3.3.7 The Definition of Limit of a Sequence

The definition of limit of a sequence, studied in calculus, uses both quantifiers $\forall$ and $\exists$ and also if-then. We say that the limit of the sequence $a_n$ as $n$ goes to infinity equals $L$ and write

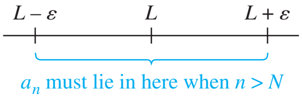
$$\lim_{n\to\infty} a_n = L$$

if, and only if, the values of $a_n$ become *arbitrarily* close to $L$ as $n$ gets larger and larger without bound. More precisely, this means that given any positive number $\varepsilon$, we can find

---

*This can be deduced from the properties of the real numbers given in Appendix A. Because $x$ is positive, $0 < x$. Add $x$ to both sides to obtain $x < 2x$. Then $0 < x < 2x$. Now multiply all parts of the inequality by the positive number 1/2. This does not change the direction of the inequality, so $0 < x/2 < x$.

an integer $N$ such that whenever $n$ is larger than $N$, the number $a_n$ sits between $L - \varepsilon$ and $L + \varepsilon$ on the number line.



Symbolically:

$$\forall \varepsilon > 0, \exists \text{ an integer } N \text{ such that } \forall \text{ integer } n,$$
$$\text{if } n > N \text{ then } L - \varepsilon < a_n < L + \varepsilon.$$

Considering the logical complexity of this definition, it is no wonder that many students find it hard to understand. ■

## Ambiguous Language

The drawing in Figure 3.3.3 is a famous example of visual ambiguity. When you look at it for a while, you will probably see either a silhouette of a young woman wearing a large hat or an elderly woman with a large nose. Whichever image first pops into your mind, try to see how the drawing can be interpreted in the other way. (*Hint:* The mouth of the elderly woman is the necklace on the young woman.)

Chronicle/Alamy Stock Photo

**FIGURE 3.3.3**

Once most people see one of the images, it is difficult for them to perceive the other. So it is with ambiguous language. Once you interpreted the sentence at the beginning of this section in one way, it may have been hard for you to see that it could be understood in the other way. Perhaps you had difficulty even though the two possible meanings were explained, just as many people have difficulty seeing the second interpretation for the drawing even when they are told what to look for.

Although statements written informally may be open to multiple interpretations, we cannot determine their truth or falsity without interpreting them one way or another. Therefore, we have to use context to try to ascertain their meaning as best we can.

## Negations of Statements with More Than One Quantifier

You can use the same rules to negate statements with several quantifiers that you used to negate simpler quantified statements. Recall that

$$\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x).$$

and

$$\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x).$$

Thus

$$\begin{aligned}\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) &\equiv \exists x \text{ in } D \text{ such that } \sim(\exists y \text{ in } E \text{ such that } P(x, y))\\ &\equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)\end{aligned}$$

Similarly,

$$\begin{aligned}\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) &\equiv \forall x \text{ in } D, \sim(\forall y \text{ in } E, P(x, y))\\ &\equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)\end{aligned}$$

These facts are summarized as follows:

### Negations of Statements with Two Different Quantifiers

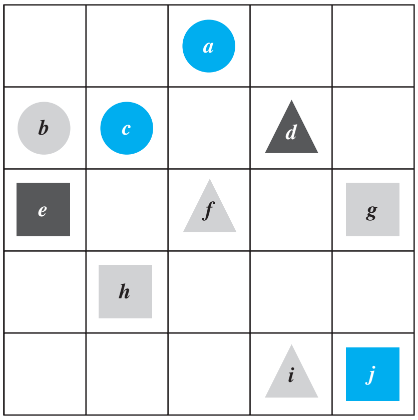
$$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$$

$$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$$

**Example 3.3.8**

### Negating Statements in a Tarski World

Refer to the Tarski world of Figure 3.3.1, which is reprinted here for reference.



Write a negation for each of the following statements, and determine which is true, the given statement or its negation.

a. For every square $x$, there is a circle $y$ such that $x$ and $y$ have the same color.

b. There is a triangle $x$ such that for every square $y$, $x$ is to the right of $y$.

### Solution

a. *First version of negation:* $\exists$ a square $x$ such that $\sim(\exists$ a circle $y$ such that $x$ and $y$ have the same color).

*Final version of negation:* $\exists$ a square $x$ such that $\forall$ circle $y$, $x$ and $y$ do not have the same color.

The negation is true. Square $e$ is black and no circle in this Tarski world is black, so there is a square that does not have the same color as any circle.

b. *First version of negation:* $\forall$ triangle $x$, $\sim(\forall$ square $y$, $x$ is to the right of $y$).

*Final version of negation:* $\forall$ triangle $x$, $\exists$ a square $y$ such that $x$ is not to the right of $y$.

The negation is true because no matter what triangle is chosen, it is not to the right of square $g$ or square $j$, which are the only squares in this Tarski world. ■

## *Order of Quantifiers*

Consider the following two statements:

$\forall$ person $x$, $\exists$ a person $y$ such that $x$ loves $y$.

$\exists$ a person $y$ such that $\forall$ person $x$, $x$ loves $y$.

Note that except for the order of the quantifiers, these statements are identical. However, the first means that given any person, it is possible to find someone whom that person loves, whereas the second means that there is one amazing individual who is loved by all people. (Reread the statements carefully to verify these interpretations!) The two sentences illustrate an extremely important property about statements with two different quantifiers.

> In a statement containing both $\forall$ and $\exists$, changing the order of the quantifiers can significantly change the meaning of the statement.

Interestingly, however, if one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning. Consider the commutative property of addition of real numbers, for example:

$$\forall \text{ real number } x \text{ and } \forall \text{ real number } y,\ x + y = y + x.$$

This means the same as

$$\forall \text{ real number } y \text{ and } \forall \text{ real number } x,\ x + y = y + x.$$

Thus the property can be expressed a little less formally as

$$\forall \text{ real numbers } x \text{ and } y,\ x + y = y + x.$$

**Example 3.3.9**

### Quantifier Order in a Tarski World

Look again at the Tarski world of Figure 3.3.1. Do the following two statements have the same truth value?

a. For every square $x$ there is a triangle $y$ such that $x$ and $y$ have different colors.

b. There exists a triangle $y$ such that for every square $x$, $x$ and $y$ have different colors.

**Caution!** If a statement contains two different quantifiers, reversing their order may change the truth value of the statement to its opposite.

**Solution** Statement (a) says that if someone gives you one of the squares from the Tarski world, you can find a triangle that has a different color. This is true. If someone gives you square $g$ or $h$ (which are gray), you can use triangle $d$ (which is black); if someone gives you square $e$ (which is black), you can use either triangle $f$ or $i$ (which are gray); and if someone gives you square $j$ (which is blue), you can use triangle $d$ (which is black) or triangle $f$ or $i$ (which are gray).

Statement (b) says that there is one particular triangle in the Tarski world that has a different color from every one of the squares in the world. This is false. Two of the triangles are gray, but they cannot be used to show the truth of the statement because the Tarski world contains gray squares. The only other triangle is black, but it cannot be used either because there is a black square in the Tarski world.

Thus one of the statements is true and the other is false, and so they have opposite truth values. ■

## Formal Logical Notation

In some areas of computer science, logical statements are expressed in purely symbolic notation. The notation involves using predicates to describe all properties of variables and omitting the words *such that* in existential statements. (When you try to figure out the meaning of a formal statement, however, it is helpful to think the words *such that* to yourself each time they are appropriate.) The formalism also depends on the following facts:

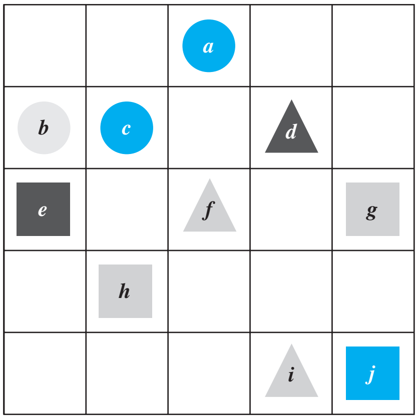
"$\forall x$ in $D$, $P(x)$" can be written as "$\forall x\ (x \text{ in } D \rightarrow P(x))$," and
"$\exists x$ in $D$ such that $P(x)$" can be written as "$\exists x\ (x \text{ in } D \wedge P(x))$."

We illustrate the use of these facts in Example 3.3.10.

**Example 3.3.10**

### Formalizing Statements in a Tarski World

Consider once more the Tarski world of Figure 3.3.1:



Let Triangle($x$), Circle($x$), and Square($x$) mean "$x$ is a triangle," "$x$ is a circle," and "$x$ is a square"; let Blue($x$), Gray($x$), and Black($x$) mean "$x$ is blue," "$x$ is gray," and "$x$ is black"; let RightOf($x$, $y$), Above($x$, $y$), and SameColorAs($x$, $y$) mean "$x$ is to the right of $y$," "$x$ is above $y$," and "$x$ has the same color as $y$"; and use the notation $x = y$ to denote the predicate "$x$ is equal to $y$." Let the common domain $D$ of all variables be the set of all the objects in the Tarski world. Use formal logical notation to write each of the following statements, and write a formal negation for each statement.

a. For every circle $x$, $x$ is above $f$.

b. There is a square $x$ such that $x$ is black.

c. For every circle $x$, there is a square $y$ such that $x$ and $y$ have the same color.

d. There is a square $x$ such that for every triangle $y$, $x$ is to the right of $y$.

### Solution

a. *Statement:* $\forall x(\text{Circle}(x) \to \text{Above}(x, f))$

*Negation:* $\sim(\forall x\, (\text{Circle}(x) \to \text{Above}(x, f)))$

$\equiv \exists x \sim(\text{Circle}(x) \to \text{Above}(x, f))$

by the law for negating a ∀ statement

$\equiv \exists x(\text{Circle}(x) \wedge \sim\text{Above}(x, f))$

by the law of negating an if-then statement

b. *Statement:* $\exists x(\text{Square}(x) \wedge \text{Black}(x))$

*Negation:* $\sim(\exists x(\text{Square}(x) \wedge \text{Black}(x)))$

$\equiv \forall x \sim(\text{Square}(x) \wedge \text{Black}(x))$

by the law for negating a ∃ statement

$\equiv \forall x(\sim\text{Square}(x) \vee \sim\text{Black}(x))$

by De Morgan's law

c. *Statement:* $\forall x(\text{Circle}(x) \to \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

*Negation:* $\sim(\forall x(\text{Circle}(x) \to \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

$\equiv \exists x \sim(\text{Circle}(x) \to \exists y(\text{Square}(y) \wedge \text{SameColor}(x, y)))$

by the law for negating a ∀ statement

$\equiv \exists x(\text{Circle}(x) \wedge \sim(\exists y(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

by the law for negating an if-then statement

$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim(\text{Square}(y) \wedge \text{SameColor}(x, y))))$

by the law for negating a ∃ statement

$\equiv \exists x(\text{Circle}(x) \wedge \forall y(\sim\text{Square}(y) \vee \sim\text{SameColor}(x, y)))$

by De Morgan's law

d. *Statement:* $\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \to \text{RightOf}(x, y)))$

*Negation:* $\sim(\exists x(\text{Square}(x) \wedge \forall y(\text{Triangle}(y) \to \text{RightOf}(x, y))))$

$\equiv \forall x\sim(\text{Square}(x) \wedge \forall y(\text{Triangle}(x) \to \text{RightOf}(x, y)))$

by the law for negating a ∃ statement

$\equiv \forall x(\sim\text{Square}(x) \vee \sim(\forall y(\text{Triangle}(y) \to \text{RightOf}(x, y))))$

by De Morgan's law

$\equiv \forall x(\sim\text{Square}(x) \vee \exists y(\sim(\text{Triangle}(y) \to \text{RightOf}(x, y))))$

by the law for negating a ∀ statement

$\equiv \forall x(\sim\text{Square}(x) \vee \exists y(\text{Triangle}(y) \wedge \sim\text{RightOf}(x, y)))$

by the law for negating an if-then statement

■

The disadvantage of the fully formal notation is that because it is complex and somewhat remote from intuitive understanding, when we use it, we may make errors that go unrecognized. The advantage, however, is that operations, such as taking negations, can be made completely mechanical and programmed on a computer. Also, when we become comfortable with formal manipulations, we can use them to check our intuition, and then we can use our intuition to check our formal manipulations. Formal logical notation is used in branches of computer science such as artificial intelligence, program verification, and automata theory and formal languages.

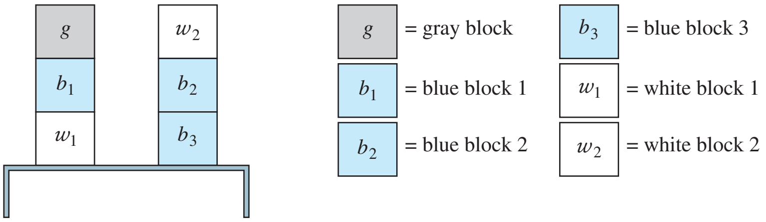
Taken together, the symbols for quantifiers, variables, predicates, and logical connectives make up what is known as the **language of first-order logic**. Even though this language is simpler in many respects than the language we use every day, learning it requires the same kind of practice needed to acquire any foreign language.

## *Prolog*

The programming language Prolog (short for *pro*gramming in *log*ic) was developed in France in the 1970s by A. Colmerauer and P. Roussel to help programmers working in the field of artificial intelligence. A simple Prolog program consists of a set of statements describing some situation together with questions about the situation. Built into the language are search and inference techniques needed to answer the questions by deriving the answers from the given statements. This frees the programmer from the necessity of having to write separate programs to answer each type of question. Example 3.3.11 gives a very simple example of a Prolog program.

### Example 3.3.11 A Prolog Program

Consider the following picture, which shows colored blocks stacked on a table.



**Note** Different Prolog implementations follow different conventions as to how to represent constant, variable, and predicate names and forms of questions and answers. The conventions used here are similar to those of Edinburgh Prolog.

The following are statements in Prolog that describe this picture and ask two questions about it.

isabove($g$, $b_1$)
isabove($b_1$, $w_1$)
isabove($w_2$, $b_2$)
isabove($b_2$, $b_3$)
color($g$, gray)
color($b_1$, blue)
color($b_2$, blue)
color($b_3$, blue)
color($w_1$, white)
color($w_2$, white)
isabove($X$, $Z$) if isabove($X$, $Y$) and isabove($Y$, $Z$)
1. ?color($b_1$, blue)
2. ?isabove($X$, $w_1$)

The statements "isabove($g$, $b_1$)" and "color($g$, gray)" are to be interpreted as "$g$ is above $b_1$" and "$g$ is colored gray." The statement "isabove($X$, $Z$) if isabove($X$, $Y$) and isabove($Y$, $Z$)" is to be interpreted as "For all $X$, $Y$, and $Z$, if $X$ is above $Y$ and $Y$ is above $Z$, then $X$ is above $Z$." Statement 1

?color($b_1$, blue)

asks whether block $b_1$ is colored blue. Prolog answers this by writing

Yes.

Statement 2

$$?\text{isabove}(X, w_1)$$

asks for which blocks $X$ the predicate "$X$ is above $w_1$" is true. Prolog answers by giving a list of all such blocks. In this case, the answer is

$$X = b_1, X = g.$$

Note that Prolog can find the solution $X = b_1$ by merely searching the original set of given facts. However, Prolog must *infer* the solution $X = g$ from the following statements:

isabove($g$, $b_1$),
isabove($b_1$, $w_1$),
isabove($X$, $Z$) if isabove($X$, $Y$) and isabove($Y$, $Z$).

Write the answers Prolog would give if the following questions were added to the program above.

a. ?isabove($b_2$, $w_1$) b. ?color($w_1$, $X$) c. ?color($X$, blue)

**Solution**

a. The question means "Is $b_2$ above $w_1$?"; so the answer is "No."

b. The question means "For what colors $X$ is the predicate '$w_1$ is colored $X$' true?"; so the answer is "$X$ = white."

c. The question means "For what blocks is the predicate '$X$ is colored blue' true?"; so the answer is "$X = b_1$," "$X = b_2$," and "$X = b_3$." ■

## TEST YOURSELF

1. To establish the truth of a statement of the form "$\forall x$ in $D$, $\exists y$ in $E$ such that $P(x, y)$," you imagine that someone has given you an element $x$ from $D$ but that you have no control over what that element is. Then you need to find _____ with the property that the $x$ the person gave you together with the _____ you subsequently found satisfy _____.

2. To establish the truth of a statement of the form "$\exists x$ in $D$ such that $\forall y$ in $E$, $P(x, y)$," you need to find _____ so that no matter what _____ a person might subsequently give you, _____ will be true.

3. Consider the statement "$\forall x$, $\exists y$ such that $P(x, y)$, a property involving $x$ and $y$, is true." A negation for this statement is "_____."

4. Consider the statement "$\exists x$ such that $\forall y$, $P(x, y)$, a property involving $x$ and $y$, is true." A negation for this statement is "_____."

5. Suppose $P(x, y)$ is some property involving $x$ and $y$, and suppose the statement "$\forall x$ in $D$, $\exists y$ in $E$ such that $P(x, y)$" is true. Then the statement "$\exists x$ in $D$ such that $\forall y$ in $E$, $P(x, y)$"
   **a.** is true.
   **b.** is false.
   **c.** may be true or may be false.

# EXERCISE SET 3.3

**1.** Let $C$ be the set of cities in the world, let $N$ be the set of nations in the world, and let $P(c, n)$ be "$c$ is the capital city of $n$." Determine the truth values of the following statements.

**a.** $P$(Tokyo, Japan)

**b.** $P$(Athens, Egypt)

**c.** $P$(Paris, France)

**d.** $P$(Miami, Brazil)

**2.** Let $G(x, y)$ be "$x^2 > y$." Indicate which of the following statements are true and which are false.

**a.** $G(2, 3)$

**b.** $G(1, 1)$

**c.** $G\left(\frac{1}{2}, \frac{1}{2}\right)$

**d.** $G(-2, 2)$

**3.** The following statement is true: "∀ nonzero number $x$, ∃ a real number $y$ such that $xy = 1$." For each $x$ given below, find a $y$ to make the predicate "$xy = 1$" true.

**a.** $x = 2$

**b.** $x = -1$

**c.** $x = 3/4$

**4.** The following statement is true: "∀ real number $x$, ∃ an integer $n$ such that $n > x$."* For each $x$ given below, find an $n$ to make the predicate "$n > x$" true.

**a.** $x = 15.83$

**b.** $x = 10^8$

**c.** $x = 10^{10^{10}}$

**The statements in exercises 5–8 refer to the Tarski world given in Figure 3.3.1. Explain why each is true.**

**5.** For every circle $x$ there is a square $y$ such that $x$ and $y$ have the same color.

**6.** For every square $x$ there is a circle $y$ such that $x$ and $y$ have different colors and $y$ is above $x$.

**7.** There is a triangle $x$ such that for every square $y$, $x$ is above $y$.

**8.** There is a triangle $x$ such that for every circle $y$, $y$ is above $x$.

**9.** Let $D = E = \{-2, -1, 0, 1, 2\}$. Explain why the following statements are true.

**a.** $\forall x$ in $D$, $\exists y$ in $E$ such that $x + y = 0$.

**b.** $\exists x$ in $D$ such that $\forall y$ in $E$, $x + y = y$.

**10.** This exercise refers to Example 3.3.3. Determine whether each of the following statements is true or false.

**a.** ∀ student $S$, ∃ a dessert $D$ such that $S$ chose $D$.

**b.** ∀ student $S$, ∃ a salad $T$ such that $S$ chose $T$.

**c.** ∃ a dessert $D$ such that ∀ student $S$, $S$ chose $D$.

**d.** ∃ a beverage $B$ such that ∀ student $D$, $D$ chose $B$.

**e.** ∃ an item $I$ such that ∀ student $S$, $S$ did not choose $I$.

**f.** ∃ a station $Z$ such that ∀ student $S$, ∃ an item $I$ such that $S$ chose $I$ from $Z$.

**11.** Let $S$ be the set of students at your school, let $M$ be the set of movies that have ever been released, and let $V(s, m)$ be "student $s$ has seen movie $m$." Rewrite each of the following statements without using the symbol ∀, the symbol ∃, or variables.

**a.** $\exists s \in S$ such that $V(s$, Casablanca$)$.

**b.** $\forall s \in S$, $V(s$, Star Wars$)$.

**c.** $\forall s \in S$, $\exists m \in M$ such that $V(s, m)$.

**d.** $\exists m \in M$ such that $\forall s \in S$, $V(s, m)$.

**e.** $\exists s \in S$, $\exists t \in S$, and $\exists m \in M$ such that $s \neq t$ and $V(s, m) \wedge V(t, m)$.

**f.** $\exists s \in S$ and $\exists t \in S$ such that $s \neq t$ and $\forall m \in M$, $V(s, m) \rightarrow V(t, m)$.

**12.** Let $D = E = \{-2, -1, 0, 1, 2\}$. Write negations for each of the following statements and determine which is true, the given statement or its negation.

**a.** $\forall x$ in $D$, $\exists y$ in $E$ such that $x + y = 1$.

**b.** $\exists x$ in $D$ such that $\forall y$ in $E$, $x + y = -y$.

**c.** $\forall x$ in $D$, $\exists y$ in $E$ such that $xy \geq y$.

**d.** $\exists x$ in $D$ such that $\forall y$ in $E$, $x \leq y$.

**In each of 13–19, (a) rewrite the statement in English without using the symbol ∀ or ∃ or variables and expressing your answer as simply as possible, and (b) write a negation for the statement.**

**13.** ∀ color $C$, ∃ an animal $A$ such that $A$ is colored $C$.

**14.** ∃ a book $b$ such that ∀ person $p$, $p$ has read $b$.

**15.** ∀ odd integer $n$, ∃ an integer $k$ such that $n = 2k + 1$.

**16.** ∃ a real number $u$ such that ∀ real number $v$, $uv = v$.

**17.** $\forall r \in \mathbf{Q}$, ∃ integers $a$ and $b$ such that $r = a/b$.

**18.** $\forall x \in \mathbf{R}$, ∃ a real number $y$ such that $x + y = 0$.

**19.** $\exists x \in \mathbf{R}$ such that for every real number $y$, $x + y = 0$.

**20.** Recall that reversing the order of the quantifiers in a statement with two different quantifiers may

---

*This is called the Archimedean principle because it was first formulated (in geometric terms) by the great Greek mathematician Archimedes of Syracuse, who lived from about 287 to 212 B.C.E.

change the truth value of the statement—but it does not necessarily do so. All the statements in the pairs below refer to the Tarski world of Figure 3.3.1. In each pair, the order of the quantifiers is reversed but everything else is the same. For each pair, determine whether the statements have the same or opposite truth values. Justify your answers.

**a.** (1) For every square $y$ there is a triangle $x$ such that $x$ and $y$ have different colors.
(2) There is a triangle $x$ such that for every square $y$, $x$ and $y$ have different colors.

**b.** (1) For every circle $y$ there is a square $x$ such that $x$ and $y$ have the same color.
(2) There is a square $x$ such that for every circle $y$, $x$ and $y$ have the same color.

**21.** For each of the following equations, determine which of the following statements are true:
(1) For every real number $x$, there exists a real number $y$ such that the equation is true.
(2) There exists a real number $x$, such that for every real number $y$, the equation is true.

Note that it is possible for both statements to be true or for both to be false.

**a.** $2x + y = 7$
**b.** $y + x = x + y$
**c.** $x^2 - 2xy + y^2 = 0$
**d.** $(x - 5)(y - 1) = 0$
**e.** $x^2 + y^2 = -1$

**In 22 and 23, rewrite each statement without using variables or the symbol ∀ or ∃. Indicate whether the statement is true or false.**

**22.** **a.** $\forall$ real number $x$, $\exists$ a real number $y$ such that $x + y = 0$.
**b.** $\exists$ a real number $y$ such that $\forall$ real number $x$, $x + y = 0$.

**23.** **a.** $\forall$ nonzero real number $r$, $\exists$ a real number $s$ such that $rs = 1$.
**b.** $\exists$ a real number $r$ such that $\forall$ nonzero real number $s$, $rs = 1$.

**24.** Use the laws for negating universal and existential statements to derive the following rules:

**a.** $\sim(\forall x \in D(\forall y \in E(P(x, y))))$
$$\equiv \exists x \in D(\exists y \in E(\sim P(x, y)))$$
**b.** $\sim(\exists x \in D(\exists y \in E(P(x, y))))$
$$\equiv \forall x \in D(\forall y \in E(\sim P(x, y)))$$

**Each statement in 25–28 refers to the Tarski world of Figure 3.3.1. For each, (a) determine whether the statement is true or false and justify your answer, and (b) write a negation for the statement (referring, if you wish, to the result in exercise 24).**

**25.** $\forall$ circle $x$ and $\forall$ square $y$, $x$ is above $y$.

**26.** $\forall$ circle $x$ and $\forall$ triangle $y$, $x$ is above $y$.

**27.** $\exists$ a circle $x$ and $\exists$ a square $y$ such that $x$ is above $y$ and $x$ and $y$ have different colors.

**28.** $\exists$ a triangle $x$ and $\exists$ a square $y$ such that $x$ is above $y$ and $x$ and $y$ have the same color.

**For each of the statements in 29 and 30, (a) write a new statement by interchanging the symbols ∀ and ∃, and (b) state which is true: the given statement, the version with interchanged quantifiers, neither, or both.**

**29.** $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ such that $x < y$.

**30.** $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}^-$ (the set of negative real numbers), $x > y$.

**31.** Consider the statement "Everybody is older than somebody." Rewrite this statement in the form "$\forall$ people $x$, $\exists$ ______."

**32.** Consider the statement "Somebody is older than everybody." Rewrite this statement in the form "$\exists$ a person $x$ such that $\forall$ ______."

**In 33–39, (a) rewrite the statement formally using quantifiers and variables, and (b) write a negation for the statement.**

**33.** Everybody loves somebody.

**34.** Somebody loves everybody.

**35.** Everybody trusts somebody.

**36.** Somebody trusts everybody.

**37.** Any even integer equals twice some integer.

**38.** Every action has an equal and opposite reaction.

**39.** There is a program that gives the correct answer to every question that is posed to it.

**40.** In informal speech most sentences of the form "There is ______ every ______" are intended to be understood as meaning "$\forall$ ______ $\exists$ ______," even though the existential quantifier *there is* comes before the universal quantifier *every*. Note that this interpretation applies to the following well-known sentences. Rewrite them using quantifiers and variables.

**a.** There is a sucker born every minute.

**b.** There is a time for every purpose under heaven.

**41.** Indicate which of the following statements are true and which are false. Justify your answers as best you can.

**a.** $\forall x \in \mathbf{Z}^+, \exists y \in \mathbf{Z}^+$ such that $x = y + 1$.

**b.** $\forall x \in \mathbf{Z}, \exists y \in \mathbf{Z}$ such that $x = y + 1$.

**c.** $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}, x = y + 1$.

**d.** $\forall x \in \mathbf{R}^+, \exists y \in \mathbf{R}^+$ such that $xy = 1$.

**e.** $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}$ such that $xy = 1$.

**f.** $\exists x \in \mathbf{R}$ such that $\forall y \in \mathbf{R}, x + y = y$.

**g.** $\forall x \in \mathbf{R}^+, \exists y \in \mathbf{R}^+$ such that $y < x$.

**h.** $\exists x \in \mathbf{R}^+$ such that $\forall y \in \mathbf{R}^+, x \leq y$.

**42.** Write the negation of the definition of limit of a sequence given in Example 3.3.7.

**43.** The following is the definition for $\lim_{x \to a} f(x) = L$:

For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every real number $x$, if $a - \delta < x < a + \delta$ and $x \neq a$ then

$$L - \varepsilon < f(x) < L + \varepsilon.$$

Write what it means for $\lim_{x \to a} f(x) \neq L$. In other words, write the negation of the definition.

**44.** The notation $\exists!$ stands for the words "there exists a unique." Thus, for instance, "$\exists!$ $x$ such that $x$ is prime and $x$ is even" means that there is one and only one even prime number. Which of the following statements are true and which are false? Explain.

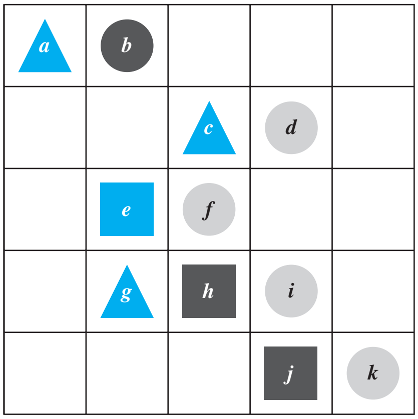
**a.** $\exists!$ real number $x$ such that $\forall$ real number $y$, $xy = y$.

**b.** $\exists!$ integer $x$ such that $1/x$ is an integer.

**c.** $\forall$ real number $x$, $\exists!$ real number $y$ such that $x + y = 0$.

***45.** Suppose that $P(x)$ is a predicate and $D$ is the domain of $x$. Rewrite the statement "$\exists!$ $x \in D$ such that $P(x)$" without using the symbol $\exists!$. (See exercise 44 for the meaning of $\exists!$.)

In 46–54, refer to the Tarski world given in Figure 3.1.1, which is shown again here for reference. The domains of all variables consist of all the objects in the Tarski world. For each statement, (a) indicate whether the statement is true or false and justify your answer, (b) write the given statement using the formal logical notation illustrated in Example 3.3.10, and (c) write a negation for the given statement using the formal logical notation of Example 3.3.10.



**46.** There is a triangle $x$ such that for every square $y$, $x$ is above $y$.

**47.** There is a triangle $x$ such that for every circle $y$, $x$ is above $y$.

**48.** For every circle $x$, there is a square $y$ such that $y$ is to the right of $x$.

**49.** For every object $x$, if $x$ is a circle then there is a square $y$ such that $y$ has the same color as $x$.

**50.** For every object $x$, if $x$ is a triangle then there is a square $y$ such that $y$ is below $x$.

**51.** There is a square $x$ such that for every triangle $y$, if $y$ is above $x$ then $y$ has the same color as $x$.

**52.** For every circle $x$ and for every triangle $y$, $x$ is to the right of $y$.

**53.** There is a circle $x$ and there is a square $y$ such that $x$ and $y$ have the same color.

**54.** There is a circle $x$ and there is a triangle $y$ such that $x$ has the same color as $y$.

Let $P(x)$ and $Q(x)$ be predicates and suppose $D$ is the domain of $x$. In 55–58, for the statement forms in each pair, determine whether (a) they have the same truth value for every choice of $P(x)$, $Q(x)$, and $D$, or (b) there is a choice of $P(x)$, $Q(x)$, and $D$ for which they have opposite truth values.

**55.** $\forall x \in D, (P(x) \wedge Q(x))$, and

$$(\forall x \in D, P(x)) \wedge (\forall x \in D, Q(x))$$

**56.** $\exists x \in D, (P(x) \wedge Q(x))$, and
$(\exists x \in D, P(x)) \wedge (\exists x \in D, Q(x))$

**57.** $\forall x \in D, (P(x) \vee Q(x))$, and
$(\forall x \in D, P(x)) \vee (\forall x \in D, Q(x))$

**58.** $\exists x \in D, (P(x) \vee Q(x))$, and
$(\exists x \in D, P(x)) \vee (\exists x \in D, Q(x))$

**In 59–61, find the answers Prolog would give if the following questions were added to the program given in Example 3.3.11.**

**59.** **a.** ?isabove($b_1$, $w_1$)
**b.** ?color($X$, white)
**c.** ?isabove($X$, $b_3$)

**60.** **a.** ?isabove($w_1$, $g$)
**b.** ?color($w_2$, blue)
**c.** ?isabove($X$, $b_1$)

**61.** **a.** ?isabove($w_2$, $b_3$)
**b.** ?color($X$, gray)
**c.** ?isabove($g$, $X$)

## ANSWERS FOR TEST YOURSELF

**1.** an element $y$ in $E$; $y$; $P(x, y)$ **2.** an element $x$ in $D$; $y$ in $E$; $P(x, y)$ **3.** $\exists x$ such that $\forall y$, the property $P(x, y)$ is false. **4.** $\forall x$, $\exists y$ such that the property $P(x, y)$ is false. **5.** The answer is (c): the truth or falsity of a statement in which the quantifiers are reversed depends on the nature of the property involving $x$ and $y$.

# 3.4 Arguments with Quantified Statements

*The only complete safeguard against reasoning ill, is the habit of reasoning well; familiarity with the principles of correct reasoning; and practice in applying those principles.* —John Stuart Mill

The rule of *universal instantiation* (in-stan-she-AY-shun) says the following:

**Universal Instantiation**

If a property is true of *everything* in a set, then it is true of *any particular* thing in the set.

Use of the words *universal instantiation* indicates that the truth of a property in a particular case follows as a special instance of its more general or universal truth. The validity of this argument form follows immediately from the definition of truth values for a universal statement. One of the most famous examples of universal instantiation is the following:

All men are mortal.
Socrates is a man.
∴ Socrates is mortal.

Universal instantiation is *the* fundamental tool of deductive reasoning. Mathematical formulas, definitions, and theorems are like general templates that are used over and over in a wide variety of particular situations. A given theorem says that such and such is true for all things of a certain type. If, in a given situation, you have a particular object of that type, then by universal instantiation, you conclude that such and such is true for that particular object. You may repeat this process 10, 20, or more times in a single proof or problem solution.

As an example of universal instantiation, suppose you are doing a problem that requires you to simplify

$$r^{k+1} \cdot r,$$

where $r$ is a particular real number and $k$ is a particular integer. You know from your study of algebra that the following universal statements are true:

1. For every real number $x$ and for all integers $m$ and $n$, $x^m \cdot x^n = x^{m+n}$.
2. For every real number $x$, $x^1 = x$.

So you proceed as follows:

$$\begin{aligned} r^{k+1} \cdot r &= r^{k+1} \cdot r^1 && \text{Step 1} \\ &= r^{(k+1)+1} && \text{Step 2} \\ &= r^{k+2} && \text{by basic algebra.} \end{aligned}$$

Here is the reasoning behind steps 1 and 2.

| | | |
|---|---|---|
| Step 1: | For every real number $x$, $x^1 = x$. | universal truth |
| | $r$ is a particular real number. | particular instance |
| | $\therefore r^1 = r$. | conclusion |
| Step 2: | For every real number $x$ and for all integers $m$ and $n$, $x^m \cdot x^n = x^{m+n}$. | universal truth |
| | $r$ is a particular real number and $k+1$ and 1 are particular integers. | particular instance |
| | $\therefore r^{k+1} \cdot r^1 = r^{(k+1)+1}$. | conclusion |

Both arguments are examples of universal instantiation.

## Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the valid form of argument called *universal modus ponens.*

### Universal Modus Ponens

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x$, if $P(x)$ then $Q(x)$. | If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true. |
| $P(a)$ for a particular $a$. | $a$ makes $P(x)$ true. |
| $\therefore Q(a)$. | $\therefore$ $a$ makes $Q(x)$ true. |

Note that the first, or major, premise of universal modus ponens could be written “All things that make $P(x)$ true make $Q(x)$ true,” in which case the conclusion would follow by universal instantiation alone. However, the if-then form is more natural to use in the majority of mathematical situations.

**Example 3.4.1**

### Recognizing Universal Modus Ponens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Is this argument valid? Why?

If an integer is even, then its square is even.

$k$ is a particular integer that is even.

$\therefore k^2$ is even.

**Solution** The major premise of this argument can be rewritten as

$$\forall x, \text{ if } x \text{ is an even integer then } x^2 \text{ is even.}$$

Let $E(x)$ be "$x$ is an even integer," let $S(x)$ be "$x^2$ is even," and let $k$ stand for a particular integer that is even. Then the argument has the following form:

$$\begin{aligned} &\forall x, \text{ if } E(x) \text{ then } S(x).\\ &E(k), \text{ for a particular } k.\\ \therefore\ &S(k). \end{aligned}$$

This argument has the form of universal modus ponens and is therefore valid. ■

## **Example 3.4.2** Drawing Conclusions Using Universal Modus Ponens

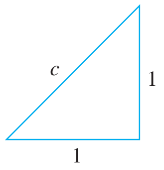
Write the conclusion that can be inferred using universal modus ponens.

If $T$ is any right triangle with hypotenuse $c$ and legs $a$ and $b$, then $c^2 = a^2 + b^2$.

The triangle shown at the right is a right triangle with both legs equal to 1 and hypotenuse $c$.

∴ ________________________________

Pythagorean Theorem



**Solution** $c^2 = 1^2 + 1^2 = 2$

Note that if you take the nonnegative square root of both sides of this equation, you obtain $c = \sqrt{2}$. This shows that there is a line segment whose length is $\sqrt{2}$. Section 4.7 contains a proof that $\sqrt{2}$ is not a rational number. ■

## *Use of Universal Modus Ponens in a Proof*

In Chapter 4 we discuss methods of proving quantified statements. Here is a proof that the sum of any two even integers is even. It makes use of the definition of even integer, namely, that an integer is *even* if, and only if, it equals twice some integer. (Or, more formally: ∀ integers $x$, $x$ is even if, and only if, ∃ an integer—say, $k$—such that $x = 2k$.)

Suppose $m$ and $n$ are particular but arbitrarily chosen even integers. Then $m = 2r$ for some integer $r$,(1) and $n = 2s$ for some integer $s$.(2) Hence

$$\begin{aligned} m + n &= 2r + 2s && \text{by substitution}\\ &= 2(r + s)^{(3)} && \text{by factoring out the 2.} \end{aligned}$$

Now $r + s$ is an integer,(4) and so $2(r + s)$ is even.(5) Thus $m + n$ is even.

The following expansion of the proof shows how each of the numbered steps is justified by arguments that are valid by universal modus ponens.

**Note** The logical principle of **existential instantiation** says that if we know or have deduced that something exists, we may give it a name. This is the principle that allows us to call the integers $r$ and $s$.

(1) If an integer is even, then it equals twice some integer.
$m$ is a particular even integer.
∴ $m$ equals twice some integer, say, $r$.

(2) If an integer is even, then it equals twice some integer.
$n$ is a particular even integer.
∴ $n$ equals twice some integer, say, $s$.

(3) If a quantity is an integer, then it is a real number.
$r$ and $s$ are particular integers.
∴ $r$ and $s$ are real numbers.

For all $a$, $b$, and $c$, if $a$, $b$, and $c$ are real numbers, then $ab + ac = a(b + c)$.
2, $r$, and $s$ are particular real numbers.
∴ $2r + 2s = 2(r + s)$.

(4) For all $u$ and $v$, if $u$ and $v$ are integers, then $u + v$ is an integer.
$r$ and $s$ are two particular integers.
$\therefore$ $r + s$ is an integer.

(5) If a number equals twice some integer, then that number is even.
$2(r + s)$ equals twice the integer $r + s$.
$\therefore$ $2(r + s)$ is even.

Of course, the actual proof that the sum of even integers is even does not explicitly contain the sequence of arguments given above. In fact, people who are good at analytical thinking are normally not even conscious that they are reasoning in this way because they have absorbed the method so completely that it has become almost as automatic as breathing.

## Universal Modus Tollens

Another crucially important rule of inference is *universal modus tollens*. Its validity results from combining universal instantiation with modus tollens. Universal modus tollens is the heart of proof of contradiction, which is one of the most important methods of mathematical argument.

### Universal Modus Tollens

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x$, if $P(x)$ then $Q(x)$. | If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true. |
| $\sim Q(a)$, for a particular $a$. | $a$ does not make $Q(x)$ true. |
| $\therefore \sim P(a)$. | $\therefore$ $a$ does not make $P(x)$ true. |

**Example 3.4.3**

### Recognizing the Form of Universal Modus Tollens

Rewrite the following argument using quantifiers, variables, and predicate symbols. Write the major premise in conditional form. Is this argument valid? Why?

All human beings are mortal.
Zeus is not mortal.
$\therefore$ Zeus is not human.

**Solution** The major premise can be rewritten as

$\forall x$, if $x$ is human then $x$ is mortal.

Let $H(x)$ be "$x$ is human," let $M(x)$ be "$x$ is mortal," and let $Z$ stand for Zeus. The argument becomes

$$\begin{aligned} &\forall x, \text{ if } H(x) \text{ then } M(x) \\ &\sim M(Z) \\ \therefore\ &\sim H(Z). \end{aligned}$$

This argument has the form of universal modus tollens and is therefore valid. ■

### Example 3.4.4 Drawing Conclusions Using Universal Modus Tollens

Write the conclusion that can be inferred using universal modus tollens.

All professors are absent-minded.
Tom Hutchins is not absent-minded.
∴ ______________________________.

**Solution** Tom Hutchins is not a professor. ■

## *Proving Validity of Arguments with Quantified Statements*

The intuitive definition of validity for arguments with quantified statements is the same as for arguments with compound statements. An argument is valid if, and only if, the truth of its conclusion follows *necessarily* from the truth of its premises. The formal definition is as follows:

**Definition**

To say that an *argument form* is **valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true. An *argument* is called **valid** if, and only if, its form is valid. It is called **sound** if, and only if, its form is valid and its premises are true.

As already noted, the validity of universal instantiation follows immediately from the definition of the truth value of a universal statement. General formal proofs of validity of arguments in the predicate calculus are beyond the scope of this book. We give the proof of the validity of universal modus ponens as an example to show that such proofs are possible and to give an idea of how they look.

Universal modus ponens asserts that

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$
$$P(a) \text{ for a particular } a.$$
$$\therefore Q(a).$$

To prove that this form of argument is valid, suppose the major and minor premises are both true. *[We must show that the conclusion "Q(a)" is also true.]* By the minor premise, $P(a)$ is true for a particular value of $a$. By the major premise and universal instantiation, the statement "If $P(a)$ then $Q(a)$" is true for that particular $a$. But by modus ponens, since the statements "If $P(a)$ then $Q(a)$" and "$P(a)$" are both true, it follows that $Q(a)$ is true also. *[This is what was to be shown.]*

The proof of validity given above is abstract and somewhat subtle. We include the proof not because we expect that you will be able to make up such proofs yourself at this stage of your study. Rather, it is intended as a glimpse of a more advanced treatment of the subject, which you can try your hand at in exercises 35 and 36 at the end of this section if you wish.

One of the paradoxes of the formal study of logic is that the laws of logic are used to prove that the laws of logic are valid!

In the next part of this section we show how you can use diagrams to analyze the validity or invalidity of arguments that contain quantified statements. Diagrams do not provide totally rigorous proofs of validity and invalidity, and in some complex settings they may even be confusing, but in many situations they are helpful and convincing.

## *Using Diagrams to Test for Validity*

Consider the statement

All integers are rational numbers.

Or, formally,

$\forall$ integer $n$, $n$ is a rational number.

Picture the set of all integers and the set of all rational numbers as disks. The truth of the given statement is represented by placing the integers disk entirely inside the rationals disk, as shown in Figure 3.4.1.

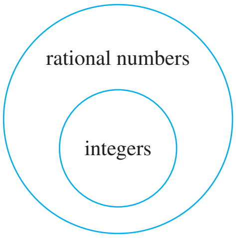


FIGURE 3.4.1

Bettmann/Getty Images

*G. W. Leibniz (1646–1716)*

Because the two statements "$\forall x \in D, Q(x)$" and "$\forall x$, if $x$ is in $D$ then $Q(x)$" are logically equivalent, both can be represented by diagrams like the foregoing.

Perhaps the first person to use diagrams like these to analyze arguments was the German mathematician and philosopher Gottfried Wilhelm Leibniz. Leibniz (LIPE-nits) was far ahead of his time in anticipating modern symbolic logic. He also developed the main ideas of the differential and integral calculus at approximately the same time as (and independently of) Isaac Newton (1642–1727).

To test the validity of an argument diagrammatically, represent the truth of both premises with diagrams. Then analyze the diagrams to see whether they necessarily represent the truth of the conclusion as well.

### Example 3.4.5 Using a Diagram to Show Validity

Use diagrams to show the validity of the following syllogism:

All human beings are mortal.
Zeus is not mortal.
$\therefore$ Zeus is not a human being.

**Solution** The major premise is pictured on the left in Figure 3.4.2 by placing a disk labeled "human beings" inside a disk labeled "mortals." The minor premise is pictured on the right in Figure 3.4.2 by placing a dot labeled "Zeus" outside the disk labeled "mortals."

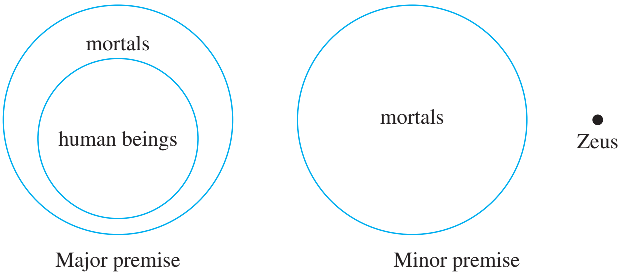


FIGURE 3.4.2

The two diagrams fit together in only one way, as shown in Figure 3.4.3.

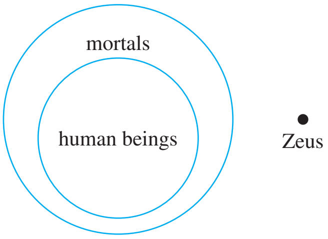


FIGURE 3.4.3

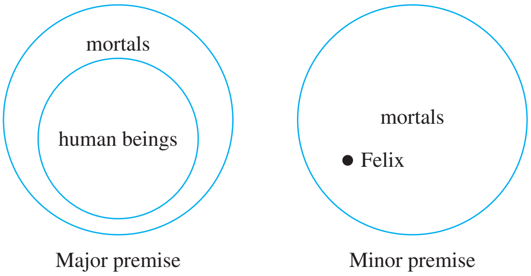
Since the Zeus dot is outside the mortals disk, it is necessarily outside the human beings disk. Thus the truth of the conclusion follows necessarily from the truth of the premises. It is impossible for the premises of this argument to be true and the conclusion false; hence the argument is valid. ■

### Example 3.4.6 Using Diagrams to Show *In*validity

Use a diagram to show the invalidity of the following argument:

All human beings are mortal.
Felix is mortal.
∴ Felix is a human being.

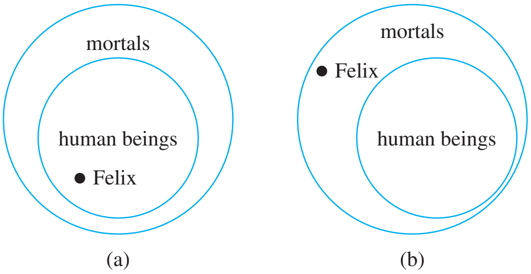
**Solution** The major and minor premises are represented diagrammatically in Figure 3.4.4.



Major premise Minor premise

FIGURE 3.4.4

All that is known is that the Felix dot is located *somewhere* inside the mortals disk. Where it is located with respect to the human beings disk cannot be determined. Either one of the situations shown in Figure 3.4.5 might be the case.



(a) (b)

FIGURE 3.4.5

The conclusion “Felix is a human being” is true in the first case but not in the second (Felix might, for example, be a cat). Because the conclusion does not necessarily follow from the premises, the argument is invalid. ■

**Caution!** Be careful when using diagrams to test for validity! For instance, in this example if you put the diagrams for the premises together to obtain only Figure 3.4.5(a) and not Figure 3.4.5(b), you would conclude erroneously that the argument was valid.

The argument of Example 3.4.6 would be valid if the major premise were replaced by its converse. But since a universal conditional statement is not logically equivalent to its converse, such a replacement cannot, in general, be made. We say that this argument exhibits the converse error.

### Converse Error (Quantified Form)

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x$, if $P(x)$ then $Q(x)$. | If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true. |
| $Q(a)$ for a particular $a$. | $a$ makes $Q(x)$ true. |
| $\therefore P(a)$. ← invalid conclusion | $\therefore$ $a$ makes $P(x)$ true. ← invalid conclusion |

The following form of argument would be valid if a conditional statement were logically equivalent to its inverse. But it is not, and the argument form is invalid. We say that it exhibits the inverse error. You are asked to show the invalidity of this argument form in the exercises at the end of this section.

### Inverse Error (Quantified Form)

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x$, if $P(x)$ then $Q(x)$. | If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true. |
| $\sim P(a)$, for a particular $a$. | $a$ does not make $P(x)$ true. |
| $\therefore \sim Q(a)$. ← invalid conclusion | $\therefore$ $a$ does not make $Q(x)$ true. ← invalid conclusion |

## Example 3.4.7 An Argument with “No”

Use diagrams to test the following argument for validity:

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
∴ This function is not a polynomial function.

**Solution** A good way to represent the major premise diagrammatically is shown in Figure 3.4.6, two disks—a disk for polynomial functions and a disk for functions with horizontal asymptotes—that do not overlap at all. The minor premise is represented by placing a dot labeled “this function” inside the disk for functions with horizontal asymptotes.

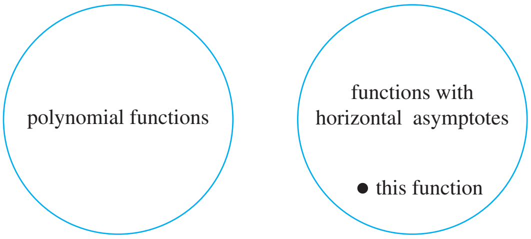


FIGURE 3.4.6

The diagram shows that "this function" must lie outside the polynomial functions disk, and so the truth of the conclusion necessarily follows from the truth of the premises. Hence the argument is valid. ■

An alternative way to solve Example 3.4.7 is to transform "No polynomial functions have horizontal asymptotes" into the equivalent statement "$\forall x$, if $x$ is a polynomial function, then $x$ does not have a horizontal asymptote." If this is done, the argument can be seen to have the form

$$\begin{aligned} &\forall x, \text{ if } P(x) \text{ then } Q(x).\\ &\sim Q(a), \text{ for a particular } a.\\ \therefore\ &\sim P(a). \end{aligned}$$

where $P(x)$ is "$x$ is a polynomial function" and $Q(x)$ is "$x$ does not have a horizontal asymptote." This is valid by universal modus tollens.

## Creating Additional Forms of Argument

Universal modus ponens and modus tollens were obtained by combining universal instantiation with modus ponens and modus tollens. In the same way, additional forms of arguments involving universally quantified statements can be obtained by combining universal instantiation with other of the valid argument forms given in Section 2.3. For instance, in Section 2.3 the argument form called transitivity was introduced:

$$\begin{aligned} &p \to q\\ &q \to r\\ \therefore\ &p \to r \end{aligned}$$

This argument form can be combined with universal instantiation to obtain the following valid argument form.

### Universal Transitivity

| *Formal Version* | *Informal Version* |
|---|---|
| $\forall x\ P(x) \to Q(x)$. | Any $x$ that makes $P(x)$ true makes $Q(x)$ true. |
| $\forall x\ Q(x) \to R(x)$. | Any $x$ that makes $Q(x)$ true makes $R(x)$ true. |
| $\therefore \forall x\ P(x) \to R(x)$. | $\therefore$ Any $x$ that makes $P(x)$ true makes $R(x)$ true. |

**Example 3.4.8**

### Evaluating an Argument for Tarski's World

The following argument refers to the kind of arrangement of objects of various types and colors described in Examples 3.1.13 and 3.3.1. Reorder and rewrite the premises to show that the conclusion follows as a valid consequence from the premises.

1. All the triangles are blue.
2. If an object is to the right of all the squares, then it is above all the circles.
3. If an object is not to the right of all the squares, then it is not blue.

$\therefore$ All the triangles are above all the circles.

**Solution** It is helpful to begin by rewriting the premises and the conclusion in if-then form:

1. $\forall x$, if $x$ is a triangle, then $x$ is blue.
2. $\forall x$, if $x$ is to the right of all the squares, then $x$ is above all the circles.

3. $\forall x$, if $x$ is not to the right of all the squares, then $x$ is not blue.

$\therefore$ $\forall x$, if $x$ is a triangle, then $x$ is above all the circles.

The goal is to reorder the premises so that the conclusion of each is the same as the hypothesis of the next. Also, the hypothesis of the argument's conclusion should be the same as the hypothesis of the first premise, and the conclusion of the argument's conclusion should be the same as the conclusion of the last premise. To achieve this goal, it may be necessary to rewrite some of the statements in contrapositive form.

In this example you can see that the first premise should remain where it is, but the second and third premises should be interchanged. Then the hypothesis of the argument is the same as the hypothesis of the first premise, and the conclusion of the argument's conclusion is the same as the conclusion of the third premise. But the hypotheses and conclusions of the premises do not quite line up. This is remedied by rewriting the third premise in contrapositive form.

1. $\forall x$, if $x$ is a triangle, then $x$ is blue.

3. $\forall x$, if $x$ is blue, then $x$ is to the right of all the squares.

2. $\forall x$, if $x$ is to the right of all the squares, then $x$ is above all the circles.

Putting 1 and 3 together and using universal transitivity gives that

4. $\forall x$, if $x$ is a triangle, then $x$ is to the right of all the squares.

And putting 4 together with 2 and using universal transitivity gives that

$\therefore$ $\forall x$, if $x$ is a triangle, then $x$ is above all the circles,

which is the conclusion of the argument. ■

## Remark on the Converse and Inverse Errors

One reason why so many people make converse and inverse errors is that the forms of the resulting arguments would be valid if the major premise were a biconditional rather than a simple conditional. And, as we noted in Section 2.2, many people tend to conflate biconditionals and conditionals.

Consider, for example, the following argument:

All the town criminals frequent the Den of Iniquity bar.
John frequents the Den of Iniquity bar.
$\therefore$ John is one of the town criminals.

The conclusion of this argument is invalid—it results from making the converse error. Therefore, it may be false even when the premises of the argument are true. This type of argument attempts unfairly to establish guilt by association.

The closer, however, the major premise comes to being a biconditional, the more likely the conclusion is to be true. If hardly anyone but criminals frequent the bar and John also frequents the bar, then it is likely (though not certain) that John is a criminal. On the basis of the given premises, it might be sensible to be suspicious of John, but it would be wrong to convict him.

A variation of the converse error is a very useful reasoning tool, provided that it is used with caution. It is the type of reasoning that is used by doctors to make medical diagnoses and by auto mechanics to repair cars. It is the type of reasoning used to generate explanations for phenomena. It goes like this:

If a statement of the form

For every $x$, if $P(x)$ then $Q(x)$

is true, and if

$Q(a)$ is true, for a particular $a$,

then check out the statement $P(a)$; it just might be true. For instance, suppose a doctor knows that

For every $x$, if $x$ has pneumonia, then $x$ has a fever and chills,
coughs deeply, and feels exceptionally tired and miserable.

And suppose the doctor also knows that

John has a fever and chills, coughs deeply,
and feels exceptionally tired and miserable.

On the basis of these data, the doctor concludes that a diagnosis of pneumonia is a strong possibility, though not a certainty. The doctor will probably attempt to gain further support for this diagnosis through laboratory testing that is specifically designed to detect pneumonia. Note that the closer a set of symptoms comes to being a necessary and sufficient condition for an illness, the more nearly certain the doctor can be of his or her diagnosis.

This form of reasoning has been named **abduction** by researchers working in artificial intelligence. It is used in certain computer programs, called expert systems, that attempt to duplicate the functioning of an expert in some field of knowledge.

## TEST YOURSELF

1. The rule of universal instantiation says that if some property is true for _____ in a domain, then it is true for _____.
2. If the first two premises of universal modus ponens are written as "If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true" and "For a particular value of $a$ _____," then the conclusion can be written as "_____."
3. If the first two premises of universal modus tollens are written as "If $x$ makes $P(x)$ true, then $x$ makes $Q(x)$ true" and "For a particular value of $a$ _____," then the conclusion can be written as "_____."
4. If the first two premises of universal transitivity are written as "Any $x$ that makes $P(x)$ true makes $Q(x)$ true" and "Any $x$ that makes $Q(x)$ true makes $R(x)$ true," then the conclusion can be written as "_____."
5. Diagrams can be helpful in testing an argument for validity. However, if some possible configurations of the premises are not drawn, a person could conclude that an argument was _____ when it was actually _____.

## EXERCISE SET 3.4

1. Let the following law of algebra be the first statement of an argument: For all real numbers $a$ and $b$,

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Suppose each of the following statements is, in turn, the second statement of the argument. Use universal instantiation or universal modus ponens to write the conclusion that follows in each case.

**a.** $a = x$ and $b = y$ are particular real numbers.
**b.** $a = f_i$ and $b = f_j$ are particular real numbers.
**c.** $a = 3u$ and $b = 5v$ are particular real numbers.
**d.** $a = g(r)$ and $b = g(s)$ are particular real numbers.
**e.** $a = \log(t_1)$ and $b = \log(t_2)$ are particular real numbers.

**Use universal instantiation or universal modus ponens to fill in valid conclusions for the arguments in 2–4.**

2. If an integer $n$ equals $2 \cdot k$ and $k$ is an integer, then $n$ is even.
0 equals $2 \cdot 0$ and 0 is an integer.
$\therefore$ _____________________.

**3.** For all real numbers $a$, $b$, $c$, and $d$, if $b \neq 0$ and $d \neq 0$ then $a/b + c/d = (ad + bc)/bd$.
$a = 2$, $b = 3$, $c = 4$, and $d = 5$ are particular real numbers such that $b \neq 0$ and $d \neq 0$.
∴ ________________.

**4.** ∀ real numbers $r$, $a$, and $b$, if $r$ is positive, then $(r^a)^b = r^{ab}$.
$r = 3$, $a = 1/2$, and $b = 6$ are particular real numbers such that $r$ is positive.
∴ ________________.

**Use universal modus tollens to fill in valid conclusions for the arguments in 5 and 6.**

**5.** All irrational numbers are real numbers.
$\frac{1}{0}$ is not a real number.
∴ ________________.

**6.** If a computer program is correct, then compilation of the program does not produce error messages.
Compilation of this program produces error messages.
∴ ________________.

**Some of the arguments in 7–18 are valid by universal modus ponens or universal modus tollens; others are invalid and exhibit the converse or the inverse error. State which are valid and which are invalid. Justify your answers.**

**7.** All healthy people eat an apple a day.
Keisha eats an apple a day.
∴ Keisha is a healthy person.

**8.** All freshmen must take a writing course.
Caroline is a freshman.
∴ Caroline must take a writing course.

**9.** If a graph has no edges, then it has a vertex of degree zero.
This graph has at least one edge.
∴ This graph does not have a vertex of degree zero.

**10.** If a product of two numbers is 0, then at least one of the numbers is 0.
For a particular number $x$, neither $(2x + 1)$ nor $(x - 7)$ equals 0.
∴ The product $(2x + 1)(x - 7)$ is not 0.

**11.** All cheaters sit in the back row.
Monty sits in the back row.
∴ Monty is a cheater.

**12.** If an 8-bit two's complement represents a positive integer, then the 8-bit two's complement starts with a 0.
The 8-bit two's complement for this integer does not start with a 0.
∴ This integer is not positive.

**13.** For every student $x$, if $x$ studies discrete mathematics, then $x$ is good at logic.
Tarik studies discrete mathematics.
∴ Tarik is good at logic.

**14.** If compilation of a computer program produces error messages, then the program is not correct.
Compilation of this program does not produce error messages.
∴ This program is correct.

**15.** Any sum of two rational numbers is rational.
The sum $r + s$ is rational.
∴ The numbers $r$ and $s$ are both rational.

**16.** If a number is even, then twice that number is even.
The number $2n$ is even, for a particular number $n$.
∴ The particular number $n$ is even.

**17.** If an infinite series converges, then the terms go to 0.
The terms of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ go to 0.
∴ The infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

**18.** If an infinite series converges, then its terms go to 0.
The terms of the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ do not go to 0.
∴ The infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ does not converge.

**19.** Rewrite the statement "No good cars are cheap" in the form "∀$x$, if $P(x)$ then $\sim Q(x)$." Indicate whether each of the following arguments is valid or invalid, and justify your answers.

**a.** No good car is cheap.
A Rimbaud is a good car.
∴ A Rimbaud is not cheap.

**b.** No good car is cheap.
A Simbaru is not cheap.
∴ A Simbaru is a good car.

**c.** No good car is cheap.
A VX Roadster is cheap.
∴ A VX Roadster is not good.

**d.** No good car is cheap.
An Omnex is not a good car.
∴ An Omnex is cheap.

**20.** **a.** Use a diagram to show that the following argument can have true premises and a false conclusion.

All dogs are carnivorous.
Aaron is not a dog.
∴ Aaron is not carnivorous.

**b.** What can you conclude about the validity or invalidity of the following argument form? Explain how the result from part (a) leads to this conclusion.

$\forall x$, if $P(x)$ then $Q(x)$.
$\sim P(a)$ for a particular $a$.
$\therefore \sim Q(a)$.

**Indicate whether the arguments in 21–27 are valid or invalid. Support your answers by drawing diagrams.**

**21.** All people are mice.
All mice are mortal.
∴ All people are mortal.

**22.** All discrete mathematics students can tell a valid argument from an invalid one.
All thoughtful people can tell a valid argument from an invalid one.
∴ All discrete mathematics students are thoughtful.

**23.** All teachers occasionally make mistakes.
No gods ever make mistakes.
∴ No teachers are gods.

**24.** No vegetarians eat meat.
All vegans are vegetarian.
∴ No vegans eat meat.

**25.** No college cafeteria food is good.
No good food is wasted.
∴ No college cafeteria food is wasted.

**26.** All polynomial functions are differentiable.
All differentiable functions are continuous.
∴ All polynomial functions are continuous.

**27.** [Adapted from Lewis Carroll.]
Nothing intelligible ever puzzles *me*.
Logic puzzles me.
∴ Logic is unintelligible.

**In exercises 28–32, reorder the premises in each of the arguments to show that the conclusion follows as a valid consequence from the premises. It may be helpful to rewrite the statements in if-then form and replace some of them by their contrapositives. Exercises 28–30 refer to the kinds of Tarski worlds discussed in Examples 3.1.13 and 3.3.1. Exercises 31 and 32 are adapted from *Symbolic Logic* by Lewis Carroll.***

**28.** 1. Every object that is to the right of all the blue objects is above all the triangles.
2. If an object is a circle, then it is to the right of all the blue objects.
3. If an object is not a circle, then it is not gray.
∴ All the gray objects are above all the triangles.

**29.** 1. All the objects that are to the right of all the triangles are above all the circles.
2. If an object is not above all the black objects, then it is not a square.
3. All the objects that are above all the black objects are to the right of all the triangles.
∴ All the squares are above all the circles.

**30.** 1. If an object is above all the triangles, then it is above all the blue objects.
2. If an object is not above all the gray objects, then it is not a square.
3. Every black object is a square.
4. Every object that is above all the gray objects is above all the triangles.
∴ If an object is black, then it is above all the blue objects.

**31.** 1. I trust every animal that belongs to me.
2. Dogs gnaw bones.
3. I admit no animals into my study unless they will beg when told to do so.
4. All the animals in the yard are mine.
5. I admit every animal that I trust into my study.
6. The only animals that are really willing to beg when told to do so are dogs.
∴ All the animals in the yard gnaw bones.

**32.** 1. When I work a logic example without grumbling, you may be sure it is one I understand.
2. The arguments in these examples are not arranged in regular order like the ones I am used to.
3. No easy examples make my head ache.
4. I can't understand examples if the arguments are not arranged in regular order like the ones I am used to.
5. I never grumble at an example unless it gives me a headache.
∴ These examples are not easy.

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*Lewis Carroll, *Symbolic Logic* (New York: Dover, 1958), pp. 118, 120, 123.

**In 33 and 34 a single conclusion follows when all the given premises are taken into consideration, but it is difficult to see because the premises are jumbled up. Reorder the premises to make it clear that a conclusion follows logically, and state the valid conclusion that can be drawn. (It may be helpful to rewrite some of the statements in if-then form and to replace some statements by their contrapositives.)**

**33.** 1. No birds except ostriches are at least 9 feet tall.
2. There are no birds in this aviary that belong to anyone but me.
3. No ostrich lives on mince pies.
4. I have no birds less than 9 feet high.

**34.** 1. All writers who understand human nature are clever.
2. No one is a true poet unless he can stir the human heart.
3. Shakespeare wrote *Hamlet*.
4. No writer who does not understand human nature can stir the human heart.
5. None but a true poet could have written *Hamlet*.

* **35.** Derive the validity of universal modus tollens from the validity of universal instantiation and modus tollens.

* **36.** Derive the validity of universal form of part (a) of the elimination rule from the validity of universal instantiation and the valid argument called elimination in Section 2.3.

## ANSWERS FOR TEST YOURSELF

**1.** all elements; any particular element in the domain (*Or:* each individual element of the domain) **2.** $P(a)$ is true; $Q(a)$ is true **3.** $Q(a)$ is false; $P(a)$ is false **4.** Any $x$ that makes $P(x)$ true makes $R(x)$ true. **5.** valid; invalid (*Or:* invalid; valid)